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Generalized Geodesic Equations From Planck Scale Deformed Dispersion Relations

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ABSTRACT

This thesis investigates how modified dispersion relations (MDRs) affect the structure of geodesic equations. MDRs generalize the standard relativistic dispersion relation by introducing momentum-dependent corrections, often arising from additional background vector fields together with the spacetime metric, as suggested by several quantum-gravity–inspired models. These modifications to the classical Hamiltonian encode possible Planck-scale effects through such background structures.

The modified geodesic equations are derived in two complementary frameworks: the Hamiltonian and the Lagrangian formalisms. In the Hamiltonian framework, MDRs are defined as level sets of the Hamilton function $H(x, p)$, and the geodesic equations are obtained from Hamilton's equations of motion. From this, a general first-order perturbed geodesic equation is derived, which was in turn used to derive the geodesic equation for an n-th order homogeneous polynomial perturbation in the momenta as a compact, ready-to-use formula.

In the Lagrangian framework, the analysis begins from the Helmholtz action, from which the corresponding Finsler (Lagrangian) function is constructed. Reinterpreting the Euler–Lagrange equations as Finsler geodesic equations provides a geometric picture of modified particle motion on the tangent bundle. The n-polynomial geodesic formula derived in this framework is then compared with its Hamiltonian counterpart, showing that the two differ only by a term that can be absorbed through reparametrization. Several specific examples are also examined to support the general results.

From this comparison, general conclusions are drawn for first-order perturbations that are homogeneous in the momenta. In particular, the conditions on the background vector field under which the modified geodesic equation reduces to the standard general-relativistic case are identified. The power of the background-field term plays a key role in determining when this reduction occurs, as higher powers lead to increasingly restrictive conditions. These results demonstrate that small variations in the perturbation structure can produce either equivalent or qualitatively distinct departures from general relativity.

CONTENTS

NOTATION AND CONVENTIONS

Throughout this thesis the following conventions are used:

- The spacetime manifold is denoted by M, with metric $g_{ab}(x)$ of Lorentzian signature $(+,-,-,-)$.
- Latin indices $a, b, c, \dots$ are used for both spacetime and momentum coordinates, with their range depending on the context (e.g. $0-3$ for spacetime indices, or $0-n$ when working with n-th order polynomials in the momenta).
- Parentheses $(\cdot)$ on indices denote symmetrization, and square brackets $[\cdot]$ denote antisymmetrization, both taken with unit weight:

$$T_{(ab)} = \tfrac{1}{2}(T_{ab} + T_{ba}), \qquad T_{[ab]} = \tfrac{1}{2}(T_{ab} - T_{ba}).$$

- $\langle p, x \rangle$ denotes the natural dual pairing (or inner product) between a covector p_a and a vector x^a.
- The Levi-Civita connection is denoted ∇_a, with Christoffel symbols $\Gamma^a{}_{bc}$.
- The Riemannian contraction is $g(p,p) := g^{ab}(x)\, p_a p_b$.
- For a background vector field $A^a(x)$, the contraction with momenta is $A(p) := A^a(x)\, p_a$ and the contaction with velocity is $A(\dot{x}) = A_a \dot{x}^a$
- Partial derivatives with respect to spacetime coordinates are written $\partial_a := \frac{\partial}{\partial x^a}$.
- Partial derivatives with respect to momentum coordinates are written $\bar{\partial}^a := \frac{\partial}{\partial p_a}, \qquad \bar{\partial}_a := g_{ab}\bar{\partial}^b$.
- Partial derivatives with respect to velocity coordinates are written as $\dot{\partial}_a := \frac{\partial}{\partial \dot{x}^a}$

I. INTRODUCTION

One of the central unresolved issues of modern theoretical physics is the reconciliation of quantum theory and gravity. While there exist many approaches for a quantum theory of gravity, for example, loop quantum gravity [1], string theory [2], causal sets [3], and group field theory [4], a complete and experimentally validated theory is yet to be discovered. Furthermore, because of the existing lack of tests, it is unknown for sure if gravity is quantum (i.e., whether general relativity can be quantized), and evidence is required to determine this. Quantum gravity phenomenology attempts to find effective signatures of the effects of quantum gravity accessible at energies far below the Planck scale [5].

A common approach through many of the quantum gravity models manifest the Planck-scale effects is modification of the standard relativistic dispersion relation [6][7]. Dispersion relations constrain a particle's momentum and determine its trajectory, ensuring physical viability. Dispersion relations can be derived from the field equations of the relevant field theory using the point-particle (or geometric-optical) limit. Furthermore, they can be reconstructed by comparing a phenomenological ansatz with data, or they appear as the Casimir operator of symmetries one imposes on relativistic point particle kinematics [6]. They also appear in quantum gravity phenomenology to describe the quantum nature of gravity that affects fields and particles. Unlike the standard relativistic relation $E^2 = p^2 + m^2$, modified dispersion relations (MDRs) contain Planck-suppressed corrections and usually result in momentum or energy-dependent velocities for particles. These corrections are extremely tiny at laboratory scales, but in astrophysical contexts they can accumulate or get amplified, producing observable consequences. A good example is in-vacuo dispersion, where the speed of photons acquires a small energy dependence [8]. Over cosmological distances, two photons of different energies emitted simultaneously—e.g., from a gamma-ray burst—would arrive on Earth with measurable time delays [9]. Time delays on the millisecond scale are within the reach of telescopes like Fermi, making astrophysical light propagation a sensitive probe of quantum gravity effects.

On the theoretical side, MDRs emerge naturally from a wide range of models. A particularly well-studied example is the κ-Poincaré algebra, a quantum deformation of the Poincaré algebra that introduces a new invariant energy scale (often near the Planck mass) [10][11].

This generates a non-trivial momentum-space geometry, with particle dynamics governed by Hamiltonian functions in curved phase space. Therefore, one must take a geometric framework on which both the spacetime curvature and non-triviality of momentum space can be encoded in order to investigate these implications in a covariant way. Hamiltonian geometry on the cotangent space of spacetime is a suitable framework for this purpose. Within this setting, dispersion relations are Hamiltonians' level sets [7]. In this picture, geometry goes beyond the spacetime metric. It also reflects the coupled phase space structure that underlies modified dispersion relations.

In parallel to the Hamiltonian approach, the Lagrangian picture offers a complementary route for the exploration of modified dispersion relations. In particular, by reformulating the dynamics in terms of generalized length functionals, one can make direct contact with the geometry of particle trajectories, usually described in a Finslerian framework. This route has been widely followed in the literature and offers valuable intuition on how MDRs deform the notion of geodesics [12]. While technically more challenging, it incorporates the dispersion relation within the variational principle naturally and provides another viewpoint that I shall return to later in this thesis.

In this thesis, I investigate how modifications of the dispersion relation due to quantum gravity effects impact the very structure of the geodesic equation. In general relativity, geodesics are trajectories for freely falling bodies and it captures the geometry of spacetime itself. Although, it is worth noting that, even within standard general relativity, the concept of massless or massive particles following an exact geodesic motion is only an approximation. Physical bodies can experience small but systematic deviations from geodesics due to radiation-reaction and self-force effects such as electromagnetic Lorentz friction and gravitational wave emission [13] [14]. These phenomena introduce additional force terms (often written schematically as F^{μ}_{self} or F^{μ}_{RR}) into the equations of motion, fundamentally altering the path of the particle relative to the pure geodesic defined by the background metric. The magnitude of these force tensors is typically suppressed by powers of Newton's constant, $G_N \sim 1/M_P^2$, making their impact extremely small in most classical regimes, yet significant when discussing quantum gravity effects, which also operate in very small scales. Although a systematic comparison is beyond the scope of this thesis, understanding the interplay between these small corrections could help clarify the origin of geodesic deviations in future research. Now by considering geodesics from MDRs, one might ask: under what conditions do particle trajectories deviate from those of general relativity? When do they remain identical to the relativistic case (up to reparametrizations)? And what are the constraints that these deviations impose on the background vector fields which encode the deviaton of the dispersion relation from GR?

To provide a response to these questions, I develop the problem in two complementary paradigms. First, in the Hamiltonian picture, I employ modified dispersion relations as level sets of a Hamiltonian function and derive the geodesic equations from Hamilton's equations of motion IV. This allows for an explicit link between modification of the dispersion relation and resulting particle dynamics. I then derive the geodesic equation for a completely general first-order perturbation of the relativistic Hamiltonian V. This result is obtained in full generality and is one of the central outcomes of the thesis. Once this equation is established, different modified geodesics can be obtained simply by substituting the specific form of the perturbation function. In this way, we can see how the structure of the geodesic equation changes depending on the type and power of the different components of the perturbation function. As part of this analysis, I also consider a perturbation function which is a n-th order polynomial form homogeneous in the momenta V B, which provides a compact formula that greatly simplifies the calculation for a wide class of possible perturbations.

Secondly, I also discuss the Lagrangian framework, where the Helmholtz action principle provides a natural way of embedding MDRs into a Lagrangian. In this picture, particle dynamics are expressed in the language of Finsler geometry, an extension of Riemannian geometry in which the metric depends on both position and velocity. Using results derived within this framework, I analyze the corresponding geodesic equation VII and compare it systematically with the Hamiltonian formulation. All of the examples considered in the Hamiltonian setting were also studied here, and by comparing the geodesic equation in these two approaches for the general n-th order polynomial case we can draw broad conclusions: the two formulations agree in structure, differing only by a term that can be absorbed through a reparametrization of the trajectory VII A.

The overall purpose of this thesis is to investigate how generic classes of MDRs warp the geodesic equation, to determine when deviations from general relativity vanish or persist, and to study their physical implications. In doing so, the work provides explicit demonstrations of quantum-gravity-induced MDRs on particle motion while contributing to the broader task of classifying MDRs within geometric and phenomenological frameworks. The results show that MDRs which appear very different at the momentum-polynomial level can sometimes yield similar geodesic structures, while in other cases they lead to genuinely new dynamical effects. This understanding clarifies which modifications are geometrically significant and points toward future directions, such as a systematic classification of MDRs alongside α–β Finsler geometries, and the exploration of observational consequences in black-hole shadows, gravitational lensing, or time-of-flight delays of high-energy photons.

II. FROM DISPERSION RELATIONS TO MODIFIED PHASE SPACE GEOMETRY

In this section, I discuss the conceptual foundation that connects modified dispersion relations (MDRs) to a geometric description of phase space. The aim is to show how MDRs, which are originally expressed as relations between energy and momentum, can be understood as deformations of the phase-space geometry encoded by the Hamiltonian. Establishing this link is essential for the later derivation of the modified geodesic equations, where these geometric ideas will be applied explicitly.

On Planck-energy scales, modified dispersion relations of particles are expected as a describe of underlying quantum gravitation effects. The modified effects can be understood geometrically as manifestations of a non-trivial curvature in momentum space [7]. This is analogous to the way the curvature in spacetime gives rise to a deviation from flat-spacetime energy-momentum relation. In different theories for quantum gravity, including κ -Poincaré symmetry or deformed special relativity, such modified effects are not related to spacetime curvature but instead are related to an intrinsic energy scale [15], for example the Planck scale, which raises the possibility that momentum space itself would become curved.

The realization suggests a change of approach: instead of working exclusively with the metric geometry of spacetime, we may adopt an approach in which momentum space and spacetime are put on an equal basis. The Hamiltonian framework naturally accommodates this because it constructs a geometry not in ordinary spacetime but in its cotangent bundle (phase space). Here, the Hamiltonian replaces the idea of the metric as something that would fix particle dynamics as well as the geometric structure of phase space. Rather than starting from an assumed background metric and deriving motion from that, we begin from an altered dispersion relation, view it as a level sets of a Hamilton function $H(x, p)$, and deduce geometry and motion together from that function.

In the case of standard relativity, when the Hamiltonian is quadratic in momenta, we find from the resulting phase-space geometry that there is a clean separation into curved spacetime and flat momentum space. But when higher-order or nonquadratic terms join the dispersion relation, which is what typically found in quantum gravity theories, this does not hold anymore. The phase space geometry is entangled so that curvature in momentum space can affect the geometry of spacetime and vice versa [7]. The key point here is that the Hamilton equations resulting from this Hamiltonian describe particle motion. Such trajectories overlap autoparallels (generalized geodesics) of phase-space geometry if and only if the Hamiltonian is homogeneous in momenta. Non-homogeneous terms will result in motion departing from free fall, adding a force-like component due to Hamiltonian intrinsic geometry. Further, momentum space geometry would naturally curve as soon as the third derivative of the Hamiltonian in terms of momenta is not zero.

Hamiltonian geometry is particularly valuable in the context of quantum gravity phenomenology. Since most quantum gravity candidates suggest that momentum space may acquire curvature or non-trivial structure (e.g., in κ -Poincaré or q-de Sitter models [7]), it becomes natural to treat momentum and position on equal footing. The Hamiltonian then defines not only how particles move but also how the geometry of spacetime and momentum space are intertwined.

In short, inhomogeneous dispersion relations naturally induce curvature in momentum space, embedding the spacetime geometry and geometry of momentum space in a non-trivial way. In what follows, we consider the κ-Poincaré model, which is a case involving inhomogeneity which makes these effects explicit and provides additional insight regarding how modified dispersion relations influence the underlying phase-space geometry as well as the physical observables derivable from it.

A. The locally κ-Poincaré Hamiltonian

A dispersion relation captures the local symmetries of phase space and defines the associated phase-space geometry, whose autoparallels are the Hamilton equations of motion. In this way, it determines how point particles move and leads to observable predictions [6, 7]. The Hamiltonian construction of the phase space geometry contains the following key features:

- The autoparallel equations of the unique torsion-free Cartan non-linear connection are the Hamilton equations of motion. They incorporate a source term that resembles a force for non-homogeneous Hamiltonians.

- The tangent spaces of phase space are separated covariantly in directions along spacetime and along momentum space by the Cartan non-linear connection, which is exclusively derived from the Hamiltonian.

- The curvature of spacetime and momentum space are defined by canonical linear connections, which are exclusively determined by the Hamiltonian and the non-linear connection. These connections generally rely on both spacetime coordinates and momenta.

- The standard Lorentzian metric geometry of spacetime with a flat momentum space is obtained for the general relativistic point particle Hamiltonian.

We now discuss a κ-deformation of the flat-spacetime Hamiltonian by introducing a modified dispersion relation given by:

$$H_\epsilon(x,p) = -\frac{4}{\ell^2}\sinh^2\left(\frac{\ell}{2}p_0\right) + e^{\ell p_0}\vec{p}^{\,2}, \tag{1}$$

where p_0 and $\vec{p}$ are, respectively, the energy and spatial momentum components of a particle, and $\ell = \epsilon^{-1}$ is the deformation parameter. In the limit $\ell \to 0$, this expression reduces to the standard relativistic Hamiltonian:

$$H(x,p) = -p_0^2 + \vec{p}^{\,2} = \eta^{\mu\nu}p_\mu p_\nu. \tag{2}$$

This κ-deformed Hamiltonian determines the effective dynamics of point particles, particularly in semiclassical regimes of quantum gravity [6]. It predicts free particle motion along straight lines in inertial coordinates, similar to special relativity. However, when this Hamiltonian is analyzed through the lens of Hamilton geometry, it reveals a crucial difference: in the ϵ-Minkowski case, the momentum space exhibits non-trivial curvature, even though the spacetime remains flat. This distinction arises as the momentum-space structure in encoded in the ϵ-deformation.

To incorporate this Hamiltonian into an arbitrary curved spacetime (M,g), we construct a generalized Hamilton function that, locally, reproduces the flat-space κ-deformed form. This construction ensures that the Hamiltonian is locally κ-Poincaré invariant in the same sense that standard general relativity is locally Lorentz invariant.

Let Z be a unit timelike vector field on the spacetime (M,g), i.e., satisfying $g(Z,Z) = -1$. This field can be interpreted as a function $Z(p) = Z^a(x)p_a$ on the cotangent bundle T^*M. The κ-Poincaré deformation in curved spacetime can then be introduced via:

$$H_{Zg}(x,p) = -\frac{4}{\ell^2}\sinh^2\left(\frac{\ell}{2}Z(p)\right) + e^{\ell Z(p)}\left(g^{-1}(p,p) + Z(p)^2\right), \tag{3}$$

Expanding this Hamiltonian in powers of ℓ, we obtain:

$$H_{Zg}(x,p) = g^{-1}(p,p) + \ell Z(p)\left(g^{-1}(p,p) + Z(p)^2\right) + \mathcal{O}(\ell^2), \tag{4}$$

where the leading term recovers the standard general relativistic Hamiltonian.

To show that this Hamiltonian is locally κ-Poincaré invariant, if one a local orthonormal frame A of the metric g such that the time direction is aligned with Z, i.e., $A_0^a \partial_a = Z$. In this frame, the energy p_0 is given by $Z(p) = A_0^a p_a$. Since A is a local frame, the metric becomes locally Minkowskian:

$$g^{ab} p_a p_b = \eta^{\mu\nu} p_\mu p_\nu = -p_0^2 + \vec{p}^{\,2}. \tag{5}$$

In this frame, the deformed Hamiltonian takes the bicrossproduct form:

$$H_{Zg}(x,p) = -\frac{4}{\ell^2}\sinh^2\left(\frac{\ell}{2}p_0\right) + e^{\ell p_0}\vec{p}^{\,2} = H_\epsilon(x,p), \tag{6}$$

demonstrating that it remains invariant under transformations generated by the κ-Poincaré algebra.

The geometry of phase space which involves both spacetime and momentum space, is encoded by this κ-deformed Hamiltonian. In this construction, momenta experience nonlinear transformations governed by the κ-Poincar´e Hopf algebra. Only those transformations that preserve the Hamiltonian (leave it invariant) are retained as physical symmetries.

There are multiple ways to define κ-deformed Hamiltonians, depending on the choice of the timelike vector field Z, which selects a local frame. Each such choice corresponds to a different basis of the κ-Poincaré Hopf algebra, potentially leading to different interaction structures or interpretations in a physical theory. Some bases preserve the classical (undeformed) translation generators but introduce deformations in their coproducts, leading to non-trivial momentum addition rules.

In this section, we focus specifically on the κ-Poincaré Hamiltonian in the bicrossproduct basis and study the motion of free particles in curved spacetime under this framework. After implementing the κ-Poincaré dispersion relation locally in a curved spacetime, we can study how particles move in phase space under the influence of the corresponding Hamiltonian. The motion is governed by the Hamilton equations of motion derived from the κ-deformed Hamiltonian. These are first-order differential equations, which are equivalent to second-order Euler-Lagrange equations when considering the associated Lagrangian.

The Hamilton equations of motion for the κ-deformed Hamiltonian H_{Zg} give the following relation between velocity and momentum:

$$\dot{x}^a = \bar{\partial}^a H_{Zg} = Z^a \left[-\frac{2}{\ell} \sinh\left(\ell Z(p)\right) + \ell e^{\ell Z(p)} \left(g^{-1}(p,p) + Z(p)^2 \right) + 2e^{\ell Z(p)} Z(p) \right] + 2e^{\ell Z(p)} g^{ab} p_b, \tag{7}$$

and the evolution of the momentum is given by:

$$\dot{p}_a = -\partial_a H_{Zg} = p_b \partial_a Z^b \left[\frac{2}{\ell} \sinh\left(\ell Z(p)\right) - \ell e^{\ell Z(p)} \left(g^{-1}(p,p) + Z(p)^2 \right) - 2e^{\ell Z(p)} Z(p) \right] - e^{\ell Z(p)} p_b p_c \partial_a g^{bc}. \tag{8}$$

To express this in a covariant way, one could use the Levi-Civita connection of the metric g, leading to a covariant form of the momentum evolution:

$$\begin{aligned}\dot{p}_a = {} & p_b \nabla_a Z^b \left[\frac{2}{\ell} \sinh\left(\ell Z(p)\right) - \ell e^{\ell Z(p)} \left(g^{-1}(p,p) + Z(p)^2 \right) - 2e^{\ell Z(p)} Z(p) \right] + 2e^{\ell Z(p)} p_c p^b \Gamma^c_{ba} \\ & + p_b \Gamma^b_{ad} Z^b \left[\frac{2}{\ell} \sinh\left(\ell Z(p)\right) - \ell e^{\ell Z(p)} \left(g^{-1}(p,p) + Z(p)^2 \right) - 2e^{\ell Z(p)} Z(p) \right].\end{aligned} \tag{9}$$

The structure of the phase space is encoded in the Hamiltonian itself, not in the metric, which is serving as a tool to check the covariance of the equations. Once Planck-scale effects (via ϵ-deformation) are present, the geometry of spacetime and momentum space become intertwined, and it is no longer possible to treat spacetime separately or assign it a standalone metric structure.

By reorganizing the above expressions, one arrives at

$$\begin{aligned} & p_c \nabla_a Z^c \left[\frac{2}{\ell} \sinh\left(\ell Z(p)\right) - \ell e^{\ell Z(p)} \left(g^{-1}(p,p) + Z(p)^2 \right) - 2e^{\ell Z(p)} Z(p) \right] = \\ & \dot{p}_a - 2e^{\ell Z(p)} p_b p_c \Gamma^c_{ab} - p_b \Gamma^b_{ad} Z^d \left[\frac{2}{\ell} \sinh\left(\ell Z(p)\right) - \ell e^{\ell Z(p)} \left(g^{-1}(p,p) + Z(p)^2 \right) - 2e^{\ell Z(p)} Z(p) \right].\end{aligned} \tag{10}$$

Since these equations are tensorial, they are covariant under coordinate transformations. This ensures that both sides transform consistently. In the limit $\ell \to 0$, the κ-deformed equations reduce to the standard geodesic equations in general relativity:

$$\dot{x}^a = 2g^{ab} p_b, \quad \dot{p}a - 2p_b p_c \Gamma^c ab = 0 \quad \Rightarrow \quad \ddot{x}^a + \Gamma^a_{bc} \dot{x}^b \dot{x}^c = \nabla_{\dot{x}} \dot{x}^a = 0. \tag{11}$$

This confirms the consistency of the deformation with the standard theory in the classical limit.

To illustrate the consequences of these ideas, we now briefly discuss two observables previously discussed in the literature in the context of κ-Poincaré dispersion relation— the Planck-scale–corrected gravitational redshift and the momentum-dependent photon sphere [6]. Both are instances of how the deformed dispersion relation alters the kinematics of particle motion by altering the dynamics. In the first case, the photon sphere acquires a momentum dependence for the particle, inheriting the sensitivity to energy in null geodesics in a phase space with deformations. In the second, gravitational redshift demonstrates a mass-dependent correction, which is a sign of violation of the weak equivalence principle at the Planck scale. Such pictures provide a concrete image of the way κ-Poincaré–type deformations affect the geometric and observational structure implied by the deformed dispersion relation.

1. *Momentum-Dependent Photon Spheres*

In general relativity, the photon sphere of the Schwarzschild black hole is a well-defined surface at constant radius $r = 3GM$, regardless of the energy or momentum of the photon. It is the surface of the unstable circular null geodesics and is of fundamental importance for determining the size and form of the black hole shadow seen by observers at large distances.

But if the dispersion relation is distorted—like in the framework of the κ-Poincaré—then the situation is qualitatively different. The distorted dispersion relation adds momentum-dependent corrections to the equations of motion. Consequently, the position of circular photon orbits is no longer universal but is rather dependent upon the angular momentum and energy of the photon.

In the κ-Poincaré model investigated in the article [6] [16], the Hamiltonian describing the spherically symmetric spacetime for a massless particle is given as

$$H_\ell(x,p) = -\frac{2}{\ell^2}\sinh^2\Big(\frac{\ell}{2}\sqrt{a(r)}\,p_t\Big) + \frac{1}{2}e^{\ell\sqrt{a(r)}\,p_t}\left(\frac{p_r^2}{a(r)} + \frac{w^2}{r^2}\right) \tag{12}$$

$$\tag{13}$$

p_t is canonical momentum conjugate to the time coordinate t p_r is the radial momentum $w = \sqrt{p_\theta^2 + \sin^{-2}\theta, p_\phi^2}$ represents the total angular momentum, $a = (1 - r_s/r)^{-1}\ \ell$ is the deformation parameter for the Planck scale.

Setting the radial momentum p_r to 0 (computing the circular orbits) and requiring the Hamiltonian to be equal to 0 results in the relationship between p_t, r, and w. Expanding this constraint perturbatively in ℓ, the radius of the photon sphere is found to become momentum dependent, as has been derived in equauation (52) of [6]:

$$r_\gamma(w) = \frac{3}{2}r_s + \ell \cdot \frac{w}{6} + \mathcal{O}(\ell^2), \tag{14}$$

The equation demonstrates the result of photons having greater angular momentum w circulating at larger radii compared to those having lower w, thus 'stretching' the photon sphere along a small radius interval. This is a direct consequence of the violation of Lorentz invariance at high energies and the manifestation of the curved momentum space created by the distorted dispersion relation.

This has significant observational implications. As the external edge of the shadow of a black hole is the unstable circular photon orbits (i.e., the photon sphere), such a momentum-dependent deviation of the orbits would cause energy-dependent distortions of the observed black hole shadow. Specifically, photons of various frequencies may contribute differently to the silhouette, resulting in measurable deviations from the expected predictions of general relativity if the scale of deformation ℓ is not too tiny. Although current measurements, such as those made using the Event Horizon Telescope (EHT), lack the resolution to identify such Planck-scale corrections now, this theoretical prediction offers a conceptually and geometrically sound avenue through which we could test the effects of quantum gravity.

2. Planck-Scale Corrections to Gravitational Redshift

An important observable where the dispersion relation modifications introduce the signature of quantum gravity is gravitational redshift. Redshift in general relativity arises from the difference in gravitational potential between the two observers, and for static observers in Schwarzschild spacetime, the frequency shift of the signal is solely given by the metric component $g_{tt}(r)$. With the introduction of MDRs, especially those respecting the κ-Poincaré symmetry, the redshift is sensitive to more than the spacetime geometry itself.

The model developed in these papers [6] [17] builds up a modified Hamiltonian for the static, spherically symmetric spacetime that obeys the local κ-Poincaré-type dispersion relation. The redshift is deduced from the difference between the energies of the photon (massless particle) observed by two observers at radii r_1 and r_2. Each observer's measurement of the energy includes projecting the momentum of the photon onto their respective 4-velocity:

$$E_i = -p_a u_i^a \tag{15}$$

Here u_i^a is the four-velocity of the observer i and p_a the covariant momentum of the photon.

The most important consequence is that, because the dispersion relation is modified, the redshift formula acquires corrections suppressed at the Planck scale, as has been established in equauation (56) of [6].

$$z + 1 = \frac{E_1}{E_2} = \sqrt{\frac{1 - r_s/r_2}{1 - r_s/r_1}} \left(1 + \frac{\ell^2}{8}(m_1^2 - m_2^2) + \mathcal{O}(\ell^4)\right), \tag{16}$$

r_1, r_2 are the radial positions of the two observers m_1, m_2 are their respective rest masses ℓ is the deformation parameter (usually varied to $\sim 1/E_{\text{Planck}}$)

Here one sees the striking aspect: the redshift is no longer solely dependent on the spacetime metric but also on the rest masses of the observers. This breaks the principle of equivalence to leading order ℓ^2, where, in regular GR, the gravitational redshift between two static observers is entirely independent of the observers' internal properties. Here, observers of different masses would measure ever so slightly different frequency shifts for the same photon—which is a prospect prohibited in classical general relativity.

At the level of physics, this constitutes a tangible pathway to experimentally test the violation of Lorentz symmetry and the effects of quantum gravity in high-precision frequency comparison measurements, most relevantly those between atomic clocks of various compositions at different gravitational potentials. Experiments along this line are already approaching the high-precision domain and may eventually become sensitive enough to test (or constrain) such Planck-scale effects of this kind. Additionally, because the deviation is proportional to the difference of the squares of the masses, it is possible to think of test experiments where the masses of the clocks are intentionally varied to optimize the Planck-suppressed correction term. Although the impact is incredibly small, suppressed by $\ell^2 \simeq 1/E^2_{\text{Planck}}$ - this is the only instance where the redshift is sensitive to the properties of the particles, and the phenomenology of the quantum structure of spacetime opens up to us.

III. GEODESIC EQUATIONS IN LAGRANGIAN AND HAMILTONIAN MECHANICS

The goal of this section is to give a quick overview of how geodesic equations occur in the Lagrangian and Hamiltonian formalisms. Having treated in previous sections the phase-space geometry of modified dispersion relations, the κ-Poincaré Hamiltonian, and a few related observables, now it is natural to come back to the basic mechanics. Here I review the simple definitions and structures of the Lagrangian and Hamiltonian frameworks and show how the geodesic equations come out in each case. In the Hamiltonian framework, the geodesics simply emerge due to Hamilton's equations of motion, and we will study this construction in preparation for generalization to deformed Hamiltonians.

In the Lagrangian formalism, the geodesic equation is a consequence of a variational principle, and in part II of this thesis, which begins with section VI, I will reformulate modified dispersion relations in terms of a Finsler function, find the corresponding Lagrangian, and study explicit examples of modified geodesics within that formalism. This introduction therefore serves a two-fold purpose: it describes the connection between the two mechanics formalisms, and it is preparation for the lengthy derivations of modified geodesic equations presented later.

A. Lagrangian Mechanics and Geodesic Equations

Let X be an n-dimensional smooth manifold (configuration space) and $L : \mathbb{R} \times TX \to \mathbb{R}$ a smooth Lagrangian function on its tangent bundle. For fixed $a < b \in \mathbb{R}$ and boundary conditions $x_a, x_b \in X$, define the path space

$$\mathcal{P} = \{q \in C^\infty([a,b], X) \mid q(a) = x_a,\ q(b) = x_b\}. \tag{17}$$

The action functional $S : \mathcal{P} \to \mathbb{R}$ is given by

$$S[q] = \int_a^b L(t, q(t), \dot{q}(t))dt. \tag{18}$$

Here, t is the parameter along the path $q(t)$, typically interpreted as coordinate time, and $\dot{q}(t) = \frac{dq}{dt}$ denotes its velocity.

A path $q \in \mathcal{P}$ is a critical point of S if and only if it satisfies the Euler–Lagrange equations

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) = 0, \quad i = 1, \ldots, n, \tag{19}$$

where $(q^i, \dot{q}^i)$ are local coordinates on TX.

Equivalently, in coordinate-free notation, q satisfies

$$\frac{\delta S}{\delta q} = 0, \tag{20}$$

where $\frac{\delta S}{\delta q}$ denotes the functional derivative of S. We begin with the line element

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \tag{21}$$

where $g_{\alpha\beta}$ is the metric with $\alpha, \beta = 0, 1, 2, 3$. Also, we are using the Einstein summation convention in which we sum over repeated indices which occur as a subscript and superscript pair. In order to find the geodesic equation, we use the variational principle which states that freely falling test particles follow a path between two fixed points in spacetime which extremizes the proper time, τ.

The proper time is defined by $d\tau^2 = -ds^2$. (We are assuming that $c = 1$.) So, formally, we have

$$\tau_{AB} = \int_A^B \sqrt{-ds^2} = \int_A^B \sqrt{-g_{\alpha\beta} dx^\alpha dx^\beta} \tag{22}$$

In order to write this as an integral that we can compute, we consider a parametrized worldline, $x^\alpha = x^\alpha(\sigma)$, where the parameter $\sigma = 0$ at point A and $\sigma = 1$ at point B. Then, we write

$$\tau_{AB} = \int_0^1 \left[-g_{\alpha\beta} \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma}\right]^{1/2} d\sigma \equiv \int_0^1 L\left[\frac{dx^\alpha}{d\sigma}, x^\alpha\right] d\sigma. \tag{23}$$

Here we have introduced the Lagrangian, $L\left[\frac{dx^\alpha}{d\sigma}, x^\alpha\right]$.
We note also that

$$L = \frac{d\tau}{d\sigma} \tag{24}$$

Therefore, for functions $f = f(\tau(\sigma))$, we have

$$\frac{df}{d\sigma} = \frac{df}{d\tau}\frac{d\tau}{d\sigma} = L\frac{df}{d\tau} \tag{25}$$

We will use this later to change derivatives with respect to our arbitrary parameter σ to derivatives with respect to the proper time, τ.

Using variational methods as seen in classical dynamics, we obtain the Euler-Lagrange equations in the form

$$-\frac{d}{d\sigma}\left(\frac{\partial L}{\partial\left(dx^\gamma/d\sigma\right)}\right) + \frac{\partial L}{\partial x^\gamma} = 0. \tag{26}$$

We carefully compute these derivatives for the general metric. First we find

$$\begin{aligned}\frac{\partial L}{\partial x^\gamma} &= -\frac{1}{2L}\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma}\\ &= -\frac{L}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\end{aligned} \tag{27}$$

The σ derivatives have been converted to τ derivatives.

Now we compute

$$\begin{aligned}\frac{\partial L}{\partial\left(dx^\gamma/d\sigma\right)} &= -\frac{1}{2L}g_{\alpha\beta}\left(\frac{dx^\beta}{d\sigma}\delta_{\alpha\gamma} + \frac{dx^\alpha}{d\sigma}\delta_{\beta\gamma}\right)\\ &= -\frac{1}{2L}\left(g_{\gamma\beta}\frac{dx^\beta}{d\sigma} + g_{\alpha\gamma}\frac{dx^\alpha}{d\sigma}\right)\\ &= -\frac{1}{L}g_{\alpha\gamma}\frac{dx^\alpha}{d\sigma}.\end{aligned} \tag{28}$$

In the last step we used the symmetry of the metric and the fact that α and β are dummy indices.

We differentiate the last result to obtain

$$\begin{aligned}-\frac{d}{d\sigma}\left(\frac{\partial L}{\partial\left(dx^\gamma/d\sigma\right)}\right) &= \frac{d}{d\sigma}\left(\frac{1}{L}g_{\alpha\gamma}\frac{dx^\alpha}{d\sigma}\right)\\ &= L\frac{d}{d\tau}\left(g_{\alpha\gamma}\frac{dx^\alpha}{d\tau}\right)\\ &= L\left[g_{\alpha\gamma}\frac{d^2x^\alpha}{d\tau^2} + \frac{1}{2}\left(\frac{\partial g_{\alpha\gamma}}{\partial x^\beta} + \frac{\partial g_{\gamma\beta}}{\partial x^\alpha}\right)\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\right].\end{aligned} \tag{29}$$

Again, we have used the symmetry of the metric and the reindexing of repeated indices. Also, we have eliminated appearances of L by changing to derivatives with respect to the proper time.

So far we have found that

$$\begin{aligned} 0 &= -\frac{d}{d\sigma}\left(\frac{\partial L}{\partial\left(dx^{\gamma}/d\sigma\right)}\right) + \frac{\partial L}{\partial x^{\gamma}} \\ &= L\left[g_{\alpha\gamma}\frac{d^2x^{\alpha}}{d\tau^2} + \frac{1}{2}\left(\frac{\partial g_{\alpha\gamma}}{\partial x^{\beta}} + \frac{\partial g_{\gamma\beta}}{\partial x^{\alpha}}\right)\frac{dx^{\alpha}}{d\tau}\frac{dx^{\beta}}{d\tau}\right] - \frac{L}{2}\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}}\frac{dx^{\alpha}}{d\tau}\frac{dx^{\beta}}{d\tau} \end{aligned} \tag{30}$$

Rearranging the terms on the right hand side we have

$$\begin{aligned} g_{\alpha\gamma}\frac{d^2x^{\alpha}}{d\tau^2} &= \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}}\frac{dx^{\alpha}}{d\tau}\frac{dx^{\beta}}{d\tau} - \frac{1}{2}\left(\frac{\partial g_{\alpha\gamma}}{\partial x^{\beta}} + \frac{\partial g_{\gamma\beta}}{\partial x^{\alpha}}\right)\frac{dx^{\alpha}}{d\tau}\frac{dx^{\beta}}{d\tau} \\ &= -\frac{1}{2}\left[\frac{\partial g_{\delta\gamma}}{\partial x^{\beta}} + \frac{\partial g_{\gamma\beta}}{\partial x^{\delta}} - \frac{\partial g_{\delta\beta}}{\partial x^{\gamma}}\right]\frac{dx^{\delta}}{d\tau}\frac{dx^{\beta}}{d\tau} \\ &\equiv -g_{\alpha\gamma}\Gamma^{\alpha}_{\delta\beta}\frac{dx^{\delta}}{d\tau}\frac{dx^{\beta}}{d\tau}. \end{aligned} \tag{31}$$

We have found the geodesic equation,

$$\frac{d^2x^{\alpha}}{d\tau^2} = -\Gamma^{\alpha}_{\delta\beta}\frac{dx^{\delta}}{d\tau}\frac{dx^{\beta}}{d\tau}, \tag{32}$$

where the Christoffel symbols satisfy

$$g_{\alpha\gamma}\Gamma^{\alpha}_{\delta\beta} = \frac{1}{2}\left[\frac{\partial g_{\gamma\delta}}{\partial x^{\beta}} + \frac{\partial g_{\gamma\beta}}{\partial x^{\delta}} - \frac{\partial g_{\delta\beta}}{\partial x^{\gamma}}\right]. \tag{33}$$

This is a linear system of equations for the Christoffel symbols. If the metric is diagonal in the coordinate system, then the computation is relatively simple as there is only one term on the left side of Equation (10). In general, one needs to use the matric inverse of $g_{\alpha\beta}$. Also, you should note that the Christoffel symbol is symmetric in the lower indices, $\Gamma^{\alpha}_{\delta\beta} = \Gamma^{\alpha}_{\beta\delta}$

We can solve for the Christoffel symbols by introducing the inverse of the metric, $g^{\mu\gamma}$.

Therefore, we can conclude that,

$$\Gamma^{\mu}_{\delta\beta} = \frac{1}{2}g^{\mu\gamma}\left[\frac{\partial g_{\gamma\delta}}{\partial x^{\beta}} + \frac{\partial g_{\gamma\beta}}{\partial x^{\delta}} - \frac{\partial g_{\delta\beta}}{\partial x^{\gamma}}\right]. \tag{34}$$

B. Hamiltonian Mechanics and Geodesic Equations

A central technique in analytical mechanics is the reduction of the n second-order Euler-Lagrange equations to a system of $2n$ first-order equations. This is achieved via the **Legendre transformation**.

Definition 1 (Legendre Transformation)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a strictly convex function, i.e., its Hessian matrix H_f is positive definite. The Legendre transformation of f is the function $g : \mathbb{R}^n \to \mathbb{R}$ defined by*

$$g(p) = \max_x [\langle p, x\rangle - f(x)] = \langle p, x(p)\rangle - f(x(p)), \tag{35}$$

where $x(p)$ is the unique solution to $p = \frac{\partial f}{\partial x}$, $\langle p, x\rangle$ denotes the natural dual pairing (or inner product) between a covector p_a and a vector x^a. This transformation is *involutive*: if g is the Legendre transform of f, then f is the Legendre transform of g.

Applying this concept to mechanics, we transform the Lagrangian $L(q, \dot{q}, t)$, viewed as a function of the velocities $\dot{q}$, to obtain the **Hamiltonian**.

Theorem 1. *The system of Euler-Lagrange equations is equivalent to the following system of $2n$ first-order equations, known as* ***Hamilton's equations****:*

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \tag{36}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q^i}, \tag{37}$$

where the Hamiltonian $H(p, q, t)$ is defined as the Legendre transform of the Lagrangian with respect to $\dot{q}$:

$$H(p, q, t) = \langle p, \dot{q}\rangle - L(q, \dot{q}, t), \quad \textit{with} \quad p_i = \frac{\partial L}{\partial \dot{x}^i}. \tag{38}$$

The Hamiltonian formalism provides a powerful tool for studying geodesic flows on Riemannian manifolds.

Let $(\mathcal{M}, g)$ be a Riemannian manifold. Let (x^i, p_i) be canonical coordinates on the cotangent bundle $T^*\mathcal{M}$, and the Hamiltonian is defined as:

$$H(q, p) = g^{ij}(x) p_i p_j. \tag{39}$$

Then, the Hamiltonian flow

$$\dot{x}^i = \frac{\partial H}{\partial p_i} = g^{ij} p_j, \tag{40}$$

$$\dot{p}_i = -\frac{\partial H}{\partial x^i} = -\frac{1}{2}\frac{\partial g^{jk}}{\partial x^i} p_j p_k \tag{41}$$

is equivalent to the geodesic equation on $\mathcal{M}$.

This establishes the fundamental connection between the geometric notion of a geodesic and the symplectic flow generated by a Hamiltonian on $T^*\mathcal{M}$.

We can start from the Hamiltonian of a free particle and derive the geodesic equation on $\mathcal{M}$ directly from Hamilton's equations.

Then by differentiating both sides of (40) with respect to the curve parameter, we arrive at the following expression

$$\begin{aligned}\ddot{x}^a &= \frac{d}{d\tau}\left(\bar{\partial}^a H\left(x, p\right)\right)\\ &= \dot{x}^b\left(\partial_b \bar{\partial}^a H\right) + \dot{p}_b\left(\bar{\partial}^b \bar{\partial}^a H\right)\\ &= \dot{x}^b\left(\partial_b \bar{\partial}^a H\right) - \partial_b H\left(\bar{\partial}^b \bar{\partial}^a H\right)\end{aligned} \tag{42}$$

Therefore, by substituting the expressions of the derivatives of the Hamiltonian, we expect to get the geodesic equation.

$$\bar{\partial}^a H = 2g^{ac}(x) p_c = \dot{x}^a \tag{43}$$

$$\bar{\partial}^b \bar{\partial}^a H = 2g^{ab}(x) \tag{44}$$

$$\partial_a H = \partial_a g^{cd}\, p_c p_d = -2p^c\, \Gamma^d_{bc}\, p_d \qquad \text{[See Appendix (155)]} \tag{45}$$

$$\begin{aligned}\partial_b \bar{\partial}^a H &= \partial_b(2\, g^{ac} p_c) = 2\, p_c \partial_b g^{ac}\\ &= -2\, p_c\left(g^{ci}\Gamma^a_{bi} + g^{ai}\Gamma^c_{bi}\right)\end{aligned} \tag{46}$$

We now substitute the explicit expressions of the derivatives into equation (42),

$$\begin{aligned}\ddot{x}^a &= \dot{x}^b\left[-2p_c\Big(g^{ci}\Gamma^a_{bi} + g^{ai}\Gamma^c_{bi}\Big)\right] - \Big(-2p^c\, \Gamma^d_{bc}\, p_d\Big)(2g^{ab})\\ &= -2\dot{x}^b p_c g^{ci}\Gamma^a_{bi} - 2\dot{x}^b p_c g^{ai}\Gamma^c_{bi} + 4p^c\, {\Gamma_{dc}}^a\, p^d.\end{aligned} \tag{47}$$

Now replacing p by velocities via $p_c = \frac{1}{2} g_{cm}\dot{x}^m$:

$$\ddot{x}^a = -\dot{x}^b \dot{x}^m \left(g_{cm} g^{ci}\right) \Gamma^a_{bi} - \dot{x}^b \dot{x}^m \, g^{ai} \left(g_{cm} \Gamma^c_{bi}\right) + \dot{x}^c \, \Gamma_{dc}{}^a \, \dot{x}^d,$$
$$= -\dot{x}^b \dot{x}^m \, \Gamma^a_{bm} - \dot{x}^b \dot{x}^m \, \Gamma_{mb}{}^a + \dot{x}^c \dot{x}^d \, \Gamma_{dc}{}^a.$$
$$\Rightarrow \ddot{x}^a + \Gamma^a_{bc} \, \dot{x}^b \dot{x}^c = 0 \tag{48}$$

It is generally expected and always feasible to rewrite the right-hand side of the equation (48) for a more general hamiltonian-derived from a modified dispersion relation in a form that mirrors the classic geodesic equation:

$$\ddot{x}^a + \Gamma^a_{bc} \dot{x}^b \dot{x}^c = D^a \tag{49}$$

Here, it D^a represents an additional term determined by the structure of the Hamiltonian. In standard general relativity, where the Hamiltonian is homogeneous and quadratic in the momenta ($H = \frac{1}{2} g^{ab}(x) p_a p_b$), this extra term disappears. The result is the well-known geodesic equation (48), which describes the path of a freely falling particle.

However, for more general Hamiltonians—particularly those derived from modified dispersion relations that break homogeneity in momentum—D^a doesn't vanish. Instead, it introduces an effective "force" term that captures the deviation from geodesic motion. This deviation reflects additional physical influences, such as those predicted by quantum gravity or momentum-dependent spacetime structures.

After establishing the standard general relativistic case, the thesis shifts focus to exploring a modified dispersion relation of the form

$$H(x,p) = g(p,p) + \epsilon \, h(x,p). \tag{50}$$

The first step was to study a specific example,

$$H(x,p) = g^{ab}(x) \, p_a p_b + \epsilon \, g^{ab}(x) \, p_a p_b \, A^c(x) \, p_c, \tag{51}$$

which introduces a background vector field $A^a(x)$. I explicitly computed the corresponding geodesic equation and then verified that this result can be reproduced by specifying this $h(x,p)$ into the general formula derived from Hamilton's equations. This step ensures that the general geodesic equation correctly captures the structure of modified trajectories arising from a specific MDR.

The broader aim of this thesis is to understand how more general modifications to the Hamiltonian lead to deviations from geodesic motion in general relativity. In particular, this thesis aims to study how a perturbation function, taken as an n-th order polynomial in the momenta, modifies the geodesic equation, and to analyze these modifications in both the Hamiltonian and the Lagrangian pictures. The goal is to identify the physical and geometric interpretation of the different types of modifications that arise, as well as to determine what kinds of nontrivial constraints might be required to ensure consistency with known physical observations.

IV. COMPUTING GEODESICS FOR $H = g(p,p) + \epsilon\, g(p,p)\, A(p)$

To explore how modified dispersion relations affect particle trajectories, let us consider a simple yet instructive first-order deformation of the standard relativistic Hamiltonian:

$$H = g^{ab} p_a p_b + \epsilon\, g^{ab} p_a p_b\, A^c p_c = g(p,p) + \epsilon\, g(p,p)\, A(p) \tag{52}$$

This form introduces a coupling between the momentum and a background vector field A^a, motivated by models with quantum-deformed symmetries [7]. Let us now derive the corresponding geodesic equation using Hamilton's equations and to do that we should start by computing the derivatives of the Hamiltonian and then substitute them in equation (42):

$$\dot{x}^a = \bar{\partial}^a H = 2p^a \left(1 + \epsilon A(p)\right) + \epsilon\, g(p,p)\, A^a \tag{53}$$

$$\bar{\partial}^b \bar{\partial}^a H = 2g^{ab} \left(1 + \epsilon A(p)\right) + 2\epsilon \left(p^a A^b + p^b A^a\right) \tag{54}$$

$$\dot{p}_b = \partial_b H = 2p_m p^n \Gamma^m_{bn} \left(1 + \epsilon A(p)\right) - \epsilon\, g(p,p)\, p_m \left(\nabla_a A^m - \Gamma^m_{an} A^n\right) \tag{55}$$

$$\begin{aligned}\partial_b \bar{\partial}^a H = &-2\left({\Gamma^c}_{bd} g^{ad} + {\Gamma^a}_{bd} g^{dc}\right) p_c (1 + \epsilon A(p)) + 2\epsilon\, p^a p_d \left(\nabla_b A^d - \Gamma^d_{bc} A^c\right) \\ &- 2\epsilon\, p^d {\Gamma^c}_{db} p_c A^a + \epsilon g(p,p) \left(\nabla_b A^a - \Gamma^a_{bc} A^c\right)\end{aligned} \tag{56}$$

The derivation can be simplified by working consistently to first order in ϵ so that in the undeformed case ($\epsilon = 0$) the usual relation $p_a = \frac{1}{2}\dot{x}_a$ is recovered. For nonzero ϵ, let us therefore expand the momentum as

$$p_a = \tfrac{1}{2}\dot{x}_a + \epsilon\, Q_a(x, \dot{x}) \tag{57}$$

To first order in ϵ,

$$A(p) = A^b p_b = \tfrac{1}{2} A(\dot{x}) + \epsilon\, A^b Q_b(x, \dot{x}), \qquad g(p,p) = \tfrac{1}{4} g(\dot{x}, \dot{x}) + \epsilon\, \dot{x}^b Q_b(x, \dot{x}) + \mathcal{O}(\epsilon). \tag{58}$$

Substituting (57) and (58) into (53) and keeping only $\mathcal{O}(\epsilon)$:

$$\begin{aligned}\dot{x}^a &= 2\Big(\tfrac{1}{2}\dot{x}^a + \epsilon Q^a\Big)\Big(1 + \epsilon \tfrac{1}{2} A(\dot{x})\Big) + \epsilon \Big(\tfrac{1}{4} g(\dot{x}, \dot{x})\Big) A^a \\ &= \dot{x}^a + \epsilon \Big[2Q^a + \tfrac{1}{2} A(\dot{x}), \dot{x}^a + \tfrac{1}{4} g(\dot{x}, \dot{x}), A^a\Big].\end{aligned} \tag{59}$$

Cancelling $\dot{x}^a$ on both sides and solving for Q^a, we get:

$$Q^a = -\frac{1}{8}\, g(\dot{x},\dot{x})\, A^a - \frac{1}{4}\, A(\dot{x})\, \dot{x}^a \tag{60}$$

where Q_a denotes the first-order correction.

Therefore,

$$\begin{aligned}
p_a &= \frac{1}{2}\,\dot{x}_a - \epsilon\left(\frac{1}{8}\, g(\dot{x},\dot{x})\, A_a + \frac{1}{4}\, A(\dot{x})\,\dot{x}_a\right)\\
A(p) &= A^c\, p_c = A^c\left(\frac{\dot{x}_c}{2} - \epsilon\left\{\frac{1}{8}\, g(\dot{x},\dot{x})\, A_c + \frac{1}{4}\, A(\dot{x})\,\dot{x}_c\right\}\right)\\
A(p) &= \frac{g(\dot{x},A)}{2} - \frac{\epsilon}{8}\, g(\dot{x},\dot{x})\, g(A,A) - \frac{\epsilon}{4}\, g(\dot{x},A)^2\\
&\Rightarrow \epsilon A(p) = \frac{\epsilon\, g(\dot{x},A)}{2} = \frac{\epsilon\, A(\dot{x})}{2} \qquad [\text{Ignoring } \mathcal{O}(\epsilon^2)]\\
g(p,p) &= g^{ab} p_a p_b\\
&= g^{ab}\left(\frac{1}{2}\dot{x}_a - \epsilon\left(\frac{1}{8} g(\dot{x},\dot{x}) A_a + \frac{1}{4} A(\dot{x})\dot{x}_a\right)\right)\left(\frac{1}{2}\dot{x}_b - \epsilon\left(\frac{1}{8} g(\dot{x},\dot{x}) A_b + \frac{1}{4} A(\dot{x})\dot{x}_b\right)\right)\\
&= \frac{1}{4} g(\dot{x},\dot{x}) - \frac{3\epsilon}{8} g(\dot{x},\dot{x}) A(\dot{x})
\end{aligned} \tag{61}$$

And $\epsilon g(p,p) = \frac{\epsilon}{4} g(\dot{x},\dot{x})$

Now, computing the following expressions:

$$\begin{aligned}
\dot{x}^b\left(\partial_b \bar{\partial}^a H\right) \quad &= -2\dot{x}^b\left(\Gamma^c_{bd}\, g^{ad} + \Gamma^a_{bd}\, g^{dc}\right)\left(\frac{\dot{x}_c}{2} + \epsilon Q_c\right) - \frac{\epsilon A(\dot{x})}{2}\left[2\dot{x}^b\left(\Gamma^c_{bd}\, g^{ad} + \Gamma^a_{bd}\, g^{dc}\right)\left(\frac{\dot{x}_c}{2}\right)\right]\\
&+ \frac{\epsilon}{2}\dot{x}^a\dot{x}_d\dot{x}^b\left(\nabla_b A^d - \Gamma^d_{bc} A^c\right) - \frac{\epsilon}{2}\dot{x}^d\dot{x}_c\dot{x}^b\Gamma^c_{bd} A^a + \frac{\epsilon}{4} g(\dot{x},\dot{x})\dot{x}^b\left(\nabla_b A^a - \Gamma^a_{bc} A^c\right)\\
&= -\dot{x}^b\dot{x}^c\Gamma_{cb}{}^a - \dot{x}^b\dot{x}^d\Gamma^a_{bd} + \frac{1}{4}\epsilon\; g(\dot{x},\dot{x})\,\dot{x}^b\Gamma_{cb}{}^a A^c + \frac{\epsilon}{2}\dot{x}^a\dot{x}^d\dot{x}^b\nabla_b A_d - \frac{\epsilon}{2}\dot{x}^a\dot{x}^d\dot{x}^b\Gamma_{dbc} A^c\\
&- \frac{\epsilon}{2}\dot{x}^c\dot{x}^d\dot{x}^b\Gamma_{dbc} A^a + \frac{\epsilon}{4} g(\dot{x},\dot{x})\,\dot{x}^b\nabla_b A^a
\end{aligned} \tag{62}$$

$$\begin{aligned}
\dot{p}_b\left(\bar{\partial}^b\bar{\partial}^a H\right) &= \left(2p_m p^n\Gamma^m_{bn}\left(1 + \epsilon A(p)\right) - \epsilon\, g(p,p)\, p_m\left(\nabla_b A^m - \Gamma^m_{bn} A^n\right)\right)\\
&\qquad\qquad \left(2g^{ab}\left(1 + \epsilon A(p)\right) + 2\epsilon\left(p^a A^b + p^b A^a\right)\right)\\
&= \dot{x}^m\dot{x}^n\,\Gamma_{mn}{}^a - \frac{\epsilon}{2} g(\dot{x},\dot{x})\Gamma_{mn}{}^a A^m\dot{x}^n - \epsilon\Gamma_{mn}{}^a A(\dot{x})\dot{x}^m\dot{x}^n + \epsilon\, A(\dot{x})\,\dot{x}^m\dot{x}^n\,\Gamma_{mn}{}^a\\
&+ \frac{\epsilon}{2}\dot{x}^a A^b\,\dot{x}^m\dot{x}^n\,\Gamma_{mbn} + \frac{\epsilon}{2}\dot{x}^b A^a\,\dot{x}^m\dot{x}^n\,\Gamma_{mbn} - \frac{\epsilon}{4} g(\dot{x},\dot{x})\,\dot{x}^m\,\nabla^a A_m + \frac{\epsilon}{4} g(\dot{x},\dot{x})\,\dot{x}^m\,\Gamma_{mn}{}^a\, A^n
\end{aligned} \tag{63}$$

Substituting these expressions in (42), the geodesic equation becomes,

$$\begin{aligned}\ddot{x}^a + \Gamma^a{}_{bd}\dot{x}^d\dot{x}^b &= \epsilon\left[-\frac{1}{4}g(\dot{x},\dot{x})\,\dot{x}^m\,\nabla^a A_m + \frac{1}{4}g(\dot{x},\dot{x})\,\dot{x}^b\nabla_b A^a + \frac{1}{2}\dot{x}^a\dot{x}^d\dot{x}^b\nabla_b A_d\right] \\ &= \frac{\epsilon}{4}\dot{x}^b\,\left(g(\dot{x},\dot{x})g^{ap}\big(\nabla_b A_p - \nabla_p A_b\big)\;+2\dot{x}^a\dot{x}^d\dot{x}^b\nabla_b A_d\right)\end{aligned} \tag{64}$$

A few key features of this equation should be mentioned. In this expression, all velocity terms can be factored out from the right-hand side, leaving only covariant derivatives of the background vector field, which means the resulting equation is purely tensorial on the base manifold. If the background vector field can be written as the partial derivative of a scalar function, $A_b = \partial_b f$, then the antisymmetric part $(\nabla_b A_p - \nabla_p A_b)$ vanishes. This means there exists a non-trivial choice of A_b for which the modification to the geodesic equation disappears. In such a case, the only remaining contribution is the term proportional to $\dot{x}^a$, which can be absorbed by a reparametrization of the curve. As a result, the underlying trajectories on the manifold coincide with those of general relativity.

This provides an interesting example—while not the most general solution, it illustrates how certain modified dispersion relations can reduce to the standard geodesics up to reparametrization.

A closely related structure to equation (64) appears already in standard general relativity. In the presence of an electromagnetic field, a charged test particle does not follow a geodesic; instead, its motion acquires an additional force term due to the electromagnetic field. In literature, it is written out as (see Carroll Eq.(6.122) [18])

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = \frac{e}{m}\,F^\mu{}_\nu\frac{dx^\nu}{d\tau}, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

$F_{\mu\nu}$ is the electromagnetic field-strength tensor. The structure on the right-hand side of our modified geodesic equation is similar in the sense that the antisymmetric combination $\nabla_p A_b - \nabla_b A_p$ plays the role of a field-strength tensor, whose contraction with the four-velocity introduces a force-like deviation from geodesic motion. In our case, however, the one-form A_a is not an electromagnetic gauge potential, but a background field encoding quantum-gravity–motivated modifications of the dispersion relation.

It is striking that, despite arising from a quantum gravity motivated deformation of the dispersion relation, the full equation of motion reproduces the same structural form as the electromagnetic force modified geodesic equation in curved spacetime. This parallel is emerging here from entirely different physical contexts.

In the following section, I will closely follow the same steps as in the previous derivation to obtain the geodesic equation for a more general first-order perturbation of the dispersion relation. The goal here is to arrive at a final expression where, once the form of the perturbation function $h(x,p)$ is specified, it can be directly substituted into the geodesic equation. This provides a systematic framework: for any new modified dispersion relation proposed in quantum gravity models, one can simply insert the corresponding perturbation function and immediately read off the associated modification to geodesic motion.

V. GENERAL FIRST ORDER PERTURBATIONS

Let g be a general Lorentzian metric and $h(x,p)$ be a function on the cotangent bundle T^*M of spacetime. A general first order modification of the GR dispersion relation is defined by the Hamilton function

$$H(x,p) = g(p,p) + \epsilon h(x,p). \tag{65}$$

The term $g(p,p) = g^{ab}(x)p_a p_b$ defines the GR point particle dispersion relation on a curved spacetime, ϵ is a perturbation parameter, counting the first nontrivial correction to GR, and $h(x,p)$ a perturbation function, which needs to be specified depending on the application in consideration. In the context of quantum gravity phenomenology, ϵ is usually related to the Planck length or Planck energy and $h(x,p)$ can, for example, be obtained from Planck scale MDRs, such as the κ-Poincaré dispersion relation and others.
The Hamilton equations of motion gives us:

$$\dot{x}^a = \bar{\partial}^a H = 2p^a + \epsilon\ \bar{\partial}^a h(x,p)\,, \quad p^a = g^{ab}(x)p_b \tag{66}$$

$$\dot{p}_a = -\partial_a H = -(\partial_a g^{cd})p_c p_d - \epsilon\ \partial_a h(x,p)\ = 2\Gamma_{dbc}p^d p^c - \epsilon\partial_b\, h(x,p). \tag{67}$$

Let us begin by introducing an ansatz for the momentum in terms of the velocity, similar to the one used in the previous section (57). Plugged into the first Hamilton equations of motion

$$\dot{x}^a = 2(\alpha\dot{x}^a + \epsilon\ Q^a(x,\dot{x})) + \epsilon\ \bar{\partial}^a h(x,(\alpha\dot{x} + \epsilon Q))\,. \tag{68}$$

From $\epsilon \to 0$ we find $\alpha = 1/2$. Then the leftover of equation(66) is

$$\begin{aligned} 0 &= \epsilon(2Q^a) + \epsilon\bar{\partial}^a h(x,\bar{p}(x,\dot{x})) \\ &= \epsilon(2Q^a + \bar{\partial}^a h(x,\alpha\dot{x})) \\ \Rightarrow Q^a &= -\frac{1}{2}\bar{\partial}^a h(x,\alpha\dot{x}))\,. \end{aligned} \tag{69}$$

Hence

$$p_a(x,\dot{x}) = \frac{1}{2}\dot{x}_a - \epsilon\frac{1}{2}g_{ac}(x)\bar{\partial}^c h(x,\bar{p}(x,\dot{x})))\,. \tag{70}$$

with $\bar{p}_a(x,\dot{x}) = 1/2\dot{x}_a$. The dispersion relation becomes

$$
\begin{aligned}
& g^{ab}(x)p_a(x,\dot{x})p_b(x,\dot{x}) + \epsilon h(x,p(x,\dot{x})) = m^2 \\
\Rightarrow \frac{1}{4}\Big(g(\dot{x},\dot{x}) - 2\epsilon \dot{x}_d \bar{\partial}^d h(x,\bar{p}(x,\dot{x}))\Big) &+ \epsilon h(x,\bar{p}(x,\dot{x})) = m^2
\end{aligned}
\tag{71}
$$

Now calculating x and p derivative of $\bar{\partial}^a H$ up to $\mathcal{O}(\epsilon)$): Derivative with respect to x:

$$
\begin{aligned}
\partial_b \bar{\partial}^a H &= 2(\partial_b g^{ac})p_c + \epsilon\, \partial_b \bar{\partial}^a h(x,p) \\
&= 2\left(-\Gamma^a_{bd} g^{cd} - \Gamma^c_{bd} g^{ad}\right)\left(\frac{1}{2}\dot{x}_c - \epsilon \frac{1}{2} g_{cf} \bar{\partial}^f h(x,\bar{p})\right) + \epsilon\, \partial_b \bar{\partial}^a h(x,p(x,\dot{x})) \\
&= -\Gamma^a_{bd} g^{cd}\left(\dot{x}_c - \epsilon g_{cf}\bar{\partial}^f h\right) - \Gamma^c_{bd} g^{ad}\left(\dot{x}_c - \epsilon g_{cf}\bar{\partial}^f h\right) + \epsilon\, \partial_b \bar{\partial}^a h(x,p(x,\dot{x})) \\
&= -\Gamma^a_{bd}\dot{x}^d - \Gamma_{cb}{}^a \dot{x}^c + \epsilon\left[\Gamma^a_{bd}\bar{\partial}^d h + \Gamma_{cb}{}^a \bar{\partial}^c h + \partial_b \bar{\partial}^a h(x,p(x,\dot{x}))\right]
\end{aligned}
\tag{72}
$$

Derivative with respect to p:

$$
\bar{\partial}^b \bar{\partial}^a H = 2g^{ab}(x) + \epsilon\, \bar{\partial}^b \bar{\partial}^a h(x,p(x,\dot{x})) \tag{73}
$$

Equation(67) in terms of $\dot{x}$

$$
\dot{p}_b = \frac{1}{2}\Gamma_{dbc}\dot{x}^d \dot{x}^c - \frac{\epsilon}{2}\Gamma_{dbc}\left(\dot{x}^d \bar{\partial}^c h + \dot{x}^c \bar{\partial}^d h\right) - \epsilon \partial_b\, h(x,p) \tag{74}
$$

Substituting all these results in equation(42)

$$
\begin{aligned}
\ddot{x}^a &= \dot{x}^b\left(\partial_b \bar{\partial}^a H\right) + \dot{p}_b\left(\bar{\partial}^b \bar{\partial}^a H\right) \\
&= \dot{x}^b\bigg(-\Gamma^a_{bd}\dot{x}^d - \Gamma_{cb}{}^a \dot{x}^c + \epsilon\left[\Gamma^a_{bd}\bar{\partial}^d h + \Gamma_{cb}{}^a \bar{\partial}^c h + \partial_b \bar{\partial}^a h(x,p(x,\dot{x}))\right]\bigg) \\
&+ [\frac{1}{2}\Gamma_{dbc}\dot{x}^d \dot{x}^c - \frac{\epsilon}{2}\Gamma_{dbc}\left(\dot{x}^d \bar{\partial}^c h + \dot{x}^c \bar{\partial}^d h\right) - \epsilon \partial_b\, h(x,p)](2g^{ab}(x) + \epsilon\, \bar{\partial}^b \bar{\partial}^a h(x,p(x,\dot{x}))) \\
&- \Gamma^a_{bd}\dot{x}^d \dot{x}^b - \Gamma_{cb}{}^a \dot{x}^c \dot{x}^b + \epsilon \dot{x}^b\left[\Gamma^a_{bd}\bar{\partial}^d h + \Gamma_{cb}{}^a \bar{\partial}^c h + \partial_b \bar{\partial}^a h(x,p(x,\dot{x}))\right] \\
&+ \Gamma_{dc}{}^a \dot{x}^d \dot{x}^c + \frac{\epsilon}{2}\Gamma_{dbc}\bar{\partial}^b \bar{\partial}^a h\, \dot{x}^d \dot{x}^c - \epsilon\, \Gamma_{dc}{}^a\left(\dot{x}^d \bar{\partial}^c h + \dot{x}^c \bar{\partial}^d h\right) - 2\epsilon\, g^{ab}\partial_b h(x,p) \\
&= -2\epsilon\, g^{ab}\partial_b h(x,p) + \epsilon \dot{x}^b \partial_b \bar{\partial}^a h(x,p(x,\dot{x})) + \epsilon(\Gamma^a_{dc} - \Gamma_{dc}{}^a)\dot{x}^d \bar{\partial}^c h + \frac{\epsilon}{2}\Gamma_{dbc}\bar{\partial}^b \bar{\partial}^a h\, \dot{x}^d \dot{x}^c
\end{aligned}
\tag{75}
$$

Therefore, the geodesic equation

$$
\begin{aligned}
\ddot{x}^a = &- \Gamma^a_{bd}\dot{x}^d \dot{x}^b - \Gamma_{cb}{}^a \dot{x}^c \dot{x}^b + \epsilon \dot{x}^b\left[\Gamma^a_{bd}\bar{\partial}^d h + \Gamma_{cb}{}^a \bar{\partial}^c h + \partial_b \bar{\partial}^a h(x,p(x,\dot{x}))\right] \\
&+ \Gamma_{dc}{}^a \dot{x}^d \dot{x}^c + \frac{\epsilon}{2}\Gamma_{dbc}\bar{\partial}^b \bar{\partial}^a h\, \dot{x}^d \dot{x}^c - \epsilon\, \Gamma_{dc}{}^a\left(\dot{x}^d \bar{\partial}^c h + \dot{x}^c \bar{\partial}^d h\right) - 2\epsilon\, g^{ab}\partial_b h(x,p)
\end{aligned}
$$

$$
\ddot{x}^a + \Gamma^a_{bd}\dot{x}^d \dot{x}^b = -2\epsilon\, g^{ab}\partial_b h(x,p) + \epsilon \dot{x}^b \partial_b \bar{\partial}^a h(x,p(x,\dot{x})) + \epsilon(\Gamma^a_{dc} - \Gamma_{dc}{}^a)\dot{x}^d \bar{\partial}^c h + \frac{\epsilon}{2}\Gamma_{dbc}\bar{\partial}^b \bar{\partial}^a h\, \dot{x}^d \dot{x}^c \tag{76}
$$

The terms in the modified geodesic equation can be grouped in a specific way to ensure that all non-covariant contributions involving Christoffel symbols cancel. This structure guarantees that the resulting equation is manifestly covariant. In particular, this organization reveals that the correction to the geodesic motion depends solely on covariant derivatives such as $\nabla_a A_b$.

$$\ddot{x}^a + \Gamma^a{}_{bd}\dot{x}^d\dot{x}^b = \epsilon\Bigg[- g^{ab}\left(2\partial_b h + \dot{x}_d \Gamma^d{}_{bc}\bar{\partial}^c h\right) + \left(\dot{x}^b\partial_b\bar{\partial}^a h + \frac{1}{2}\dot{x}^c\dot{x}^d\Gamma_{dbc}\bar{\partial}^b\bar{\partial}^a h + \dot{x}^b\Gamma^a{}_{bd}\bar{\partial}^d h\right)\Bigg]. \tag{77}$$

Equation (77) represents the one of the main result of this thesis. From this point onward, the procedure is straightforward: one specifies a perturbation function $h(x,p)$, computes its derivatives, and substitutes them into this general expression. Both of the bracketed terms contribute only covariant derivatives of the background vector field, independent of any further assumptions. This is precisely what makes the formula so powerful. All later examples in the Hamiltonian framework build directly on this result.

For example, let us choose $h = A^q p_q$, then

$$\epsilon\partial_b h = \epsilon p_q \partial_b A^p = \frac{\epsilon}{2}\dot{x}_p\partial_b A^p\,, \quad \epsilon\bar{\partial}^c h = A^c\,, \quad \epsilon\bar{\partial}^a\bar{\partial}^b h = 0\,, \tag{78}$$

and thus

$$\begin{aligned}
\ddot{x}^a + \Gamma^a{}_{bd}\dot{x}^d\dot{x}^b &= \epsilon\Big[- g^{ab}(\dot{x}_p\partial_b A^p + \dot{x}_d\Gamma^d{}_{bc}A^c) + (\dot{x}^b\partial_b A^a + \dot{x}^b\Gamma^a{}_{bd}A^d)\Big] \\
&= \epsilon\dot{x}^p g^{ab}\Big[- \nabla_b A_p + \nabla_p A_b\Big] \\
&= \epsilon\dot{x}^p g^{ab}\left[\nabla_p A_b - \nabla_b A_p\right]\,.
\end{aligned} \tag{79}$$

A similar structure was found already in the modified geodesic equation (64) of previous example we derived , where the equation resembled the equation of motion for a charged particle in curved spacetime in the presence of an electromagnetic field. It is therefore noteworthy that an analogous structure persists here, originating from a quantum-gravity background rather than a gauge field.

Having referred to the linear perturbation function, my first concrete example I considered was the mixed case with the perturbation function $h(x,\bar{p}) = g(\bar{p},\bar{p})\,A(\bar{p})$. Plugging in this form of h in the general first-order perturbed geodesic equation (77), we arrive at precisely the same geodesic equation (64) we had derived earlier by starting from Hamilton's equations of motion. The detailed derivation of the geodesic equation can be found in Appendix A 1, where all intermediate steps are presented. This is a nice consistency test of our formalism. From the analogy, we immediately recognize that the same pattern antisymmetric-like appears in each of these cases. In addition, there is a second term proportional to $\dot{x}^a \cdots$ in the $h(x,\bar{p}) = g(\bar{p},\bar{p})\,A(\bar{p})$ case, which can be fully absorbed by a reparametrization of the trajectory in the following way. We can change the parametrization from $\tau \to s(\tau)$ such that

$$\begin{aligned} \dot{x}(s(\tau)) &= \frac{d}{d\tau}x(s(\tau)) = \frac{dx}{ds}\frac{ds}{d\tau} = x'\dot{s}\,, \\ \ddot{x}(s(\tau)) &= x'\ddot{s} + x''\dot{s}^2\,, \end{aligned} \tag{80}$$

which turns equation((A7)) to

$$x''^a + \Gamma^a{}_{bc}x'^b x'^c = = \frac{\epsilon}{4}\dot{x}'^b \left(g(\dot{x}',\dot{x}')\big(\nabla_b A^a - \nabla^a A_b\big) \; + \dot{x}'^a\Big(2\dot{x}'^d\dot{x}'^b\nabla_b A_d - \frac{\ddot{s}}{\dot{s}^2}\Big)\right) \tag{81}$$

The last term vanishes if one choose a parametrization of the curve such that

$$\frac{\ddot{s}}{\dot{s}^2} = 2\dot{x}'^d\dot{x}'^b\nabla_b A_d \tag{82}$$

This similarity between the linear case and the $g(\bar{p},\bar{p})A(\bar{p})$ case is quite striking, since the two dispersion relations are quite distinct when written out explicitly in Minkowski spacetime. One neat way of seeing how different these MDRs are, is to work locally in an orthonormal frame of the metric adapted to the background vector field. We can start by picking a point $x \in M$ and choose a tetrad $\{e_{\hat{\mu}}\}$ with $g_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}} = \mathrm{diag}(-1,1,1,1)$, and align the time–like direction of this frame with the vector field.

$$A^a(x) = Z(x)\, e_{\hat{0}}{}^a, \qquad Z := \sqrt{-\,g(A,A)}.$$

The momentum components can be written in this frame as $p_{\hat{\mu}} = (p_{\hat{0}}, \mathbf{p})$. Then

$$g^{ab}p_a p_b = -p_{\hat{0}}^2 + |\mathbf{p}|^2, \qquad A(p) = A^a p_a = Z\, e^a{}_{\hat{0}} p_a = Z\, p_{\hat{0}}.$$

With these identities the MDRs of interest become:

- Linear $h = A(p)$:

$$-p_{\hat{0}}^2 + |\mathbf{p}|^2 + \varepsilon\, Z\, p_{\hat{0}} = 0 \quad \Rightarrow \quad p_{\hat{0}} \approx |\mathbf{p}| + \tfrac{\varepsilon Z}{2} \;\; (\text{to } O(\varepsilon)), \tag{83}$$

- Mixed cubic $h = g(p,p)\, A(p)$:

$$-p_{\hat{0}}^2 + |\mathbf{p}|^2 + \varepsilon\, (-p_{\hat{0}}^2 + |\mathbf{p}|^2)\, Z\, p_{\hat{0}} = 0, \tag{84}$$

which is cubic in $p_{\hat{0}}$ (energy) and looks algebraically very different from the linear case.

To understand this similarity more clearly, we analyze another cubic case in momenta $A(p)^3$. The detailed derivation of this example is given in Appendix A 2. In the final form of the geodesic equation(A14), one again finds the antisymmetric term already encountered in the mixed cubic case, but in addition a new contribution appears which is now proportional to A^a. Unlike in the mixed case, this A^a-term cannot be absorbed through a simple reparametrization of the trajectory. To get the necessary condition for the geodesic equation to reduce to the general relativistic case, one must instead solve the differential equation directly, setting the right-hand side of the geodesic equation equal to zero. Therefore, even though both the perturbation functions are cubic in the momenta they led very different conditions that has to be met if we don't want the geodesic equation to deviate from the general relativistic one. This prompts us to consider the more general Hamiltonian of the form $H = (g(p,p))^k\, (A(p))^l$, which will allow us to analyze systematically what powers of $g(p,p)$ or $A(p)$ contribute to specific structures in the modified geodesic equation.

A. Deriving the geodesic equation for $h(x,p) = \left(g^{bc}(x)p_b p_c\right)^k \left(A^d(x)p_d\right)^r$

In this section I derive the geodesic equation for the perturbation function $h(x,p) = \left(g^{bc}(x)p_b p_c\right)^k \left(A^d(x)p_d\right)^r$, this case generalizes all the other cases we have discussed so far. We begin by calculating the perturbation function's derivatives and then substituting the p-s in terms of $\dot{x}$-s.

The p derivative of h is given by:

$$\begin{aligned}\bar{\partial}^a h &= k\, g(p,p)^{k-1}\, (A(p))^r \left(2g^{ab}p_b\right) \;+\; r\, g(p,p)^k\, (A(p))^{r-1}\, A^a. \\ &= k\left(\tfrac{1}{4}g(\dot{x},\dot{x})\right)^{k-1}\left(\tfrac{1}{2}A(\dot{x})\right)^r \dot{x}^a + r\left(\tfrac{1}{4}g(\dot{x},\dot{x})\right)^k \left(\tfrac{1}{2}A(\dot{x})\right)^{r-1} A^a.\end{aligned} \tag{85}$$

Let us define $g(\dot{x},\dot{x}) = \mathcal{G}$ and $A(\dot{x}) = \mathcal{A}$ to make the calculation look a bit more neat.

Then

$$\bar{\partial}^a h = k\left(\tfrac{1}{4}\mathcal{G}\right)^{k-1}\left(\tfrac{1}{2}\mathcal{A}\right)^r \dot{x}^a + r\left(\tfrac{1}{4}\mathcal{G}\right)^k\left(\tfrac{1}{2}\mathcal{A}\right)^{r-1} A^a. \tag{86}$$

The x derivative:

$$\begin{aligned}\partial_b h &= -2k\, g(p,p)^{k-1} A(p)^r\, \Gamma^d_{bc} p^c p_d + r\, g(p,p)^k A(p)^{r-1}\, p_d\big(\nabla_b A^d - \Gamma^d_{bc}A^c\big)\\ &= -\tfrac{1}{2}k\left(\tfrac{1}{4}\mathcal{G}\right)^{k-1}\left(\tfrac{1}{2}\mathcal{A}\right)^r \Gamma^d_{bc}\dot{x}^c\dot{x}_d + \tfrac{1}{2}r\left(\tfrac{1}{4}\mathcal{G}\right)^k\left(\tfrac{1}{2}\mathcal{A}\right)^{r-1}\dot{x}_d(\nabla_b A^d - \Gamma^d_{bc}A^c).\end{aligned} \tag{87}$$

The p derivative of $\bar{\partial}^a h$:

$$\begin{aligned}\bar{\partial}^b\bar{\partial}^a h &= 4k(k-1)\, g(p,p)^{k-2}(A(p))^r\, g^{bd}p_d\, g^{ac}p_c + 2kr\, g(p,p)^{k-1}(A(p))^{r-1}A^b g^{ac}p_c\\ &\quad + 2k\, g(p,p)^{k-1}(A(p))^r g^{ab} + 2kr\, g(p,p)^{k-1}(A(p))^{r-1} g^{bd}p_d A^a \\ &\qquad + r(r-1)\, g(p,p)^k (A(p))^{r-2} A^b A^a.\end{aligned} \tag{88}$$

$$\begin{aligned}&= k(k-1)\left(\tfrac{1}{4}\mathcal{G}\right)^{k-2}\left(\tfrac{1}{2}\mathcal{A}\right)^r \dot{x}^a\dot{x}^b + kr\left(\tfrac{1}{4}\mathcal{G}\right)^{k-1}\left(\tfrac{1}{2}\mathcal{A}\right)^{r-1}\left(A^b\dot{x}^a + A^a\dot{x}^b\right)\\ &\quad + 2k\left(\tfrac{1}{4}\mathcal{G}\right)^{k-1}\left(\tfrac{1}{2}\mathcal{A}\right)^r g^{ab} + r(r-1)\left(\tfrac{1}{4}\mathcal{G}\right)^k\left(\tfrac{1}{2}\mathcal{A}\right)^{r-2} A^a A^b.\end{aligned} \tag{89}$$

The x derivative of $\bar{\partial}^a h$:

$$\begin{aligned}\partial_b\bar{\partial}^a h &= 2k(k-1)\, g(p,p)^{k-2}\, A(p)^r\, (\partial_b g^{cd})p_c p_d\, g^{ae}p_e + 2kr\, g(p,p)^{k-1}\, A(p)^{r-1}\, (\partial_b A^d)p_d\, g^{ae}p_e\\ &\quad + 2k\, g(p,p)^{k-1}\, A(p)^r\, (\partial_b g^{ae})p_e + kr\, g(p,p)^{k-1}\, A(p)^{r-1}\, (\partial_b g^{cd})p_c p_d\, A^a\\ &\quad + r(r-1)\, g(p,p)^k\, A(p)^{r-2}\, (\partial_b A^d)p_d\, A^a + r\, g(p,p)^k\, A(p)^{r-1}\, \partial_b A^a.\\ &= -\tfrac{1}{2}k(k-1)\left(\tfrac{1}{4}\mathcal{G}\right)^{k-2}\left(\tfrac{1}{2}\mathcal{A}\right)^r \Gamma^d_{bc}\dot{x}^c\dot{x}_d\dot{x}^a + \tfrac{1}{2}kr\left(\tfrac{1}{4}\mathcal{G}\right)^{k-1}\left(\tfrac{1}{2}\mathcal{A}\right)^{r-1}\dot{x}^a\dot{x}_d(\nabla_b A^d - \Gamma^d_{bc}A^c)\\ &\quad - 2k\left(\tfrac{1}{4}\mathcal{G}\right)^{k-1}\left(\tfrac{1}{2}\mathcal{A}\right)^r\left(\tfrac{1}{2}\Gamma^a_{bi}\dot{x}^i + \tfrac{1}{2}g^{ai}\Gamma^c_{bi}\dot{x}_c\right) - \tfrac{1}{2}kr\left(\tfrac{1}{4}\mathcal{G}\right)^{k-1}\left(\tfrac{1}{2}\mathcal{A}\right)^{r-1}\Gamma^d_{bc}\dot{x}^c\dot{x}_d A^a\\ &\quad + \tfrac{1}{2}r(r-1)\left(\tfrac{1}{4}\mathcal{G}\right)^k\left(\tfrac{1}{2}\mathcal{A}\right)^{r-2} A^a\dot{x}_d(\nabla_b A^d - \Gamma^d_{bc}A^c) + \left(\tfrac{1}{4}\mathcal{G}\right)^k\left(\tfrac{1}{2}\mathcal{A}\right)^{r-1}(\nabla_b A^a - \Gamma^a_{bc}A^c).\end{aligned} \tag{90}$$

We now substitute these components into the modified geodesic equation(77):

$$\begin{aligned}\ddot{x}^a + \Gamma^a{}_{bd}\,\dot{x}^d\dot{x}^b &= \epsilon\bigg[-r\left(\tfrac{1}{4}\mathcal{G}\right)^k\left(\tfrac{1}{2}\mathcal{A}\right)^{r-1}\dot{x}^d\nabla^a A_d + \tfrac{1}{2}kr\Big(\tfrac{1}{4}\mathcal{G}\Big)^{k-1}\Big(\tfrac{1}{2}\mathcal{A}\Big)^{r-1}\dot{x}^b\dot{x}^a\dot{x}_d\nabla_b A^d\\ &\qquad + \tfrac{1}{2}r(r-1)\Big(\tfrac{1}{4}\mathcal{G}\Big)^k\Big(\tfrac{1}{2}\mathcal{A}\Big)^{r-2}A^a\dot{x}^b\dot{x}_d\nabla_b A^d + r\Big(\tfrac{1}{4}\mathcal{G}\Big)^k\Big(\tfrac{1}{2}\mathcal{A}\Big)^{r-1}\dot{x}^d\nabla_d A^a\bigg]\\ &= \epsilon\bigg[r\Big(\tfrac{1}{4}\mathcal{G}\Big)^k\Big(\tfrac{1}{2}\mathcal{A}\Big)^{r-1}\dot{x}^d g^{ab}\Big(\nabla_d A_b - \nabla_b A_d\Big) + \tfrac{1}{2}kr\Big(\tfrac{1}{4}\mathcal{G}\Big)^{k-1}\Big(\tfrac{1}{2}\mathcal{A}\Big)^{r-1}\dot{x}^b\dot{x}^a\dot{x}^d\nabla_b A_d\\ &\qquad + \tfrac{1}{2}r(r-1)\Big(\tfrac{1}{4}g(\dot{x},\dot{x})\Big)^k\Big(\tfrac{1}{2}A(\dot{x})\Big)^{r-2}A^a\dot{x}^b\dot{x}^d\nabla_b A_d\end{aligned} \tag{91}$$

Therefore, in the more general case, the modified geodesic equation acquires three characteristic structures. First, there is an antisymmetric piece of the form $g^{ab}\Big(\nabla_d A_b - \nabla_b A_d\Big)$; this is the same tensorial structure that appeared in all earlier examples. Second, a term proportional to $\dot{x}^a$ is present; it can be removed by a reparametrization of the curve, so it does not change the underlying trajectory. Third, there is a term proportional to A^a; unlike in the mixed cubic case ($g(p,p)\,A(p)$) or the linear case, this contribution does not in general disappear under reparametrization and only vanishes when $r = 1$. This explains why the linear and the mixed cubic examples looked dynamically similar in spite of their different momentum homogeneities: both share $r = 1$. In all other cases ($r \neq 1$), one must solve the differential equation obtained by setting the right-hand side of the geodesic equation to zero to get the necessary conditions that would lead to the GR case.

This framework also generalizes naturally to modified dispersion relations of the form $(A(p))^N$, which capture the string-inspired microscopic recoil model as a special case. We recover the modified geodesic equation for $A(p)^N$ (A21), by setting $k = 0$ and $r = N$ in equation (91), which yields the modified geodesic equation

$$\ddot{x}^q + \Gamma^q{}_{ij}\dot{x}^i\dot{x}^j = \epsilon\,\frac{N}{2^{,N-1}}\,A(\dot{x})^{N-1}\dot{x}^m\left[g^{qb}\Big(\nabla_m A_b - \nabla_b A_m\Big) + \frac{(N-1)A^q\dot{x}^b}{A(\dot{x})}\,\nabla_b A_m\right]. \tag{92}$$

A concrete realization of space-time foam effects was proposed in [19], in the framework of Liouville string theory. In this model, quantum defects in the vacuum act as a refractive medium for neutral particles, leading to a modified dispersion relation for photons of the form

$$p_\gamma \simeq E_\gamma - \frac{E_\gamma^2}{M_{\rm QG}}$$

with $M_{\rm QG}$ a characteristic quantum-gravity scale, typically near the Planck mass. A key feature is that only photons, being neutral under the Standard Model gauge group, couple to these space-time fluctuations, while electrons and other charged particles remain unaffected and retain their standard Lorentz-invariant dispersion relations. This violation of the equivalence principle makes photons unique probes of such quantum-gravity-induced effects, consistent with stringent synchrotron bounds on the electron sector while leaving high-energy gamma-ray observations as the primary testing ground for linear MDRs. This structure of the modification can be cast in Hamiltonian form as a perturbation $h = (A(p))^4$ added to the general relativistic dispersion relation. More generally, one may consider higher powers, $h = (A(p))^N$, which yield Planck-suppressed deviations scaling as E^{N-2} in Minkowski space-time. To study possible deviations of photon trajectories from general relativity, we take the perturbation $h = (A(p))^N$, insert it into the geodesic equation(77), and follow similar steps of derivation that we did earlier; a detailed calculation can be found in Appendix A 3 There it is shown explicitly that the modified geodesic equation for this class of models contains all three characteristic structures identified previously: the antisymmetric term, the reparametrization term proportional to $\dot{x}^a$, and the additional term multiplied with A^a. Whether or not this last contribution survives depends on the power N: for $N = 1$ the A^a-term disappears, while for $N > 1$ it remains, as already illustrated in the $(A(p))^3$ example. In addition to this MDR, there is another string-theory–inspired MDR for D-particle backgrounds and vacuum refractive indices, characterized by $h = \left(g^{ab}p_a p_b - (A^q p_q)^2\right)^N$. I will derive the geodesic equation from this MDR in Appendix A 4, where the calculation is carried out following the same procedure we have used so far: computing all derivatives of h and substituting them into equation (77). The theoretical motivation for this model is discussed briefly in VII B, when we derive the geodesic equation for this MDR in Lagrangian framework. In principle, this case could also have been derived directly from the n-polynomial master formula, which we will be deriving in the upcoming section, since it too fits into the class of polynomial perturbations. However, we defer its treatment with n-polynomial formula to the Lagrangian picture, where we present the derivation explicitly. This allows us to see more clearly how the calculation unfolds when using the general geodesic equation instead of the compact polynomial formula, and it provides a useful comparison of the relative difficulty of working with these two approaches.

B. Geodesic equation for an MDR with a n-th order polynomial term in p

In this section, we will turn to a more general version of the perturbation function, written as

$$h(x,p) = h^{a_1 \cdots a_n}(x)\, p_{a_1} \cdots p_{a_n}, \qquad p_a = \tfrac{1}{2}\dot{x}_a, \tag{93}$$

with $h^{a_1 \cdots a_n}$ totally symmetric. Some examples of this $h(x,p)$ could be

- Polynomial in $A(p)$

$$h(x,p) = (A^a p_a)^n = (A^{a_1} p_{a_1})...(A^{a_n} p_{a_n}) = A^{a_1}...A^{a_n} p_{a_1}...p_{a_n} \Leftrightarrow h^{a_1...a_n} = A^{a_1}...A^{a_n}\,. \tag{94}$$

- A mix between a metric and a 1-form ($n = 3$ example)

$$h(x,p) = g(p,p)A(p) = g^{ab}A^c p_a p_b p_c \Leftrightarrow h^{abc} = \frac{1}{3}(g^{ab}A^c + g^{bc}A^a + g^{ac}A^b) = g^{(ab}A^{c)}\,, \tag{95}$$

where round brackets around indices denote normalized symmetrization.

This formulation allows us to derive a fully general expression modified geodesic equation for the first order perturbation to the MDR, encompassing all previous examples as special cases. In particular, it not only recovers the simpler polynomial deformations discussed earlier, but also provides a unified framework that includes more complex polynomial string-inspired models. For instance, the D-particle background model and the vacuum refractive index model. Later, in the Lagrangian picture, we would revisit the same example and we shall use the n-th order polynomial formula in Lagrangian framework to derive the geodesic equation. We would see that the n-th order polynomial formula in both frameworks ends up being nearly identical up to a parameterization. Therefore, we may obtain a fairly clear understanding of how the steps of the derivation would have appeared if we had used the n-th order polynomial geodesic formula in the Hamiltonian framework.

To derive the geodesic equation for this n-order polynomial in momenta, we follow the exactly same steps of the derivation that we had followed for the previous examples. We start by calculating the derivatives of the perturbation function h.

Derivative with respect to x:

$$\begin{aligned}\partial_b h &= (\nabla_b h^{a_1...a_n})\, p_{a_1}\cdots p_{a_n} - n\,\Gamma^c{}_{ba_1}\, h^{a_1 a_2...a_n}\, p_c p_{a_2}\cdots p_{a_n} \\ &= \left(\tfrac{1}{2}\right)^n \Big[(\nabla_b h^{a_1...a_n})\, \dot{x}_{a_1}\cdots\dot{x}_{a_n} - n\,\Gamma^c{}_{ba_1}\, h^{a_1 a_2...a_n}\, \dot{x}_c \dot{x}_{a_2}\cdots\dot{x}_{a_n}\Big].\end{aligned} \tag{96}$$

Derivative with respect to p:

$$\begin{aligned}\bar{\partial}^a h &= n\, h^{aa_2...a_n}\, p_{a_2}\cdots p_{a_n} \\ &= n\left(\tfrac{1}{2}\right)^{n-1} h^{aa_2...a_n}\, \dot{x}_{a_2}\cdots\dot{x}_{a_n}.\end{aligned} \tag{97}$$

Second derivative with respect to p:

$$\bar{\partial}^b \bar{\partial}^a h = n(n-1)\, h^{aba_3\ldots a_n}\, p_{a_3} \cdots p_{a_n}$$
$$= n(n-1)\Big(\tfrac{1}{2}\Big)^{n-2} h^{aba_3\ldots a_n}\, \dot{x}_{a_3} \cdots \dot{x}_{a_n}. \tag{98}$$

Mixed derivative (x then p):

$$\partial_b \bar{\partial}^a h = n\,(\nabla_b h^{aa_2\ldots a_n})\, p_{a_2} \cdots p_{a_n} - n\, \Gamma^a{}_{bc}\, h^{ca_2\ldots a_n}\, p_{a_2} \cdots p_{a_n}$$
$$- n(n-1)\, \Gamma^c{}_{ba_2}\, h^{aa_2\ldots a_n}\, p_c p_{a_3} \cdots p_{a_n}$$
$$= \Big(\tfrac{1}{2}\Big)^{n-1} \Big[n\,(\nabla_b h^{aa_2\ldots a_n})\, \dot{x}_{a_2} \cdots \dot{x}_{a_n} - n\, \Gamma^a{}_{bc}\, h^{ca_2\ldots a_n}\, \dot{x}_{a_2} \cdots \dot{x}_{a_n}$$
$$- n(n-1)\, \Gamma^c{}_{ba_2}\, h^{aa_2\ldots a_n}\, \dot{x}_c \dot{x}_{a_3} \cdots \dot{x}_{a_n} \Big]. \tag{99}$$

First part of equation (77)

$$2\partial_b h \ + \ \dot{x}_d \Gamma^d{}_{bc}\, \bar{\partial}^c h = 2\Big(\tfrac{1}{2}\Big)^n \Big[(\nabla_b h^{a_1\ldots a_n})\, \dot{x}_{a_1} \cdots \dot{x}_{a_n} - n\, \Gamma^c{}_{ba_1}\, h^{a_1 a_2\ldots a_n}\, \dot{x}_c \dot{x}_{a_2} \cdots \dot{x}_{a_n} \Big]$$
$$+ n\Big(\tfrac{1}{2}\Big)^{n-1} \dot{x}_d \Gamma^d{}_{bc}\, h^{ca_2\ldots a_n}\, \dot{x}_{a_2} \cdots \dot{x}_{a_n}.$$
$$= \ \Big(\tfrac{1}{2}\Big)^{n-1} (\nabla_b h^{a_1\ldots a_n})\, \dot{x}_{a_1} \cdots \dot{x}_{a_n} \tag{100}$$

And the second part:

$$\dot{x}^b \partial_b \bar{\partial}^a h + \frac{1}{2} \dot{x}^c \dot{x}^d \Gamma_{dbc} \bar{\partial}^b \bar{\partial}^a h + \dot{x}^b \Gamma^a{}_{bd} \bar{\partial}^d h$$
$$= \dot{x}^b \Bigg[n\Big(\tfrac{1}{2}\Big)^{n-1} (\nabla_b h^{aa_2\ldots a_n}) \dot{x}_{a_2} \cdots \dot{x}_{a_n} - n\Big(\tfrac{1}{2}\Big)^{n-1} \Gamma^a{}_{bc}\, h^{ca_2\ldots a_n} \dot{x}_{a_2} \cdots \dot{x}_{a_n}$$
$$- n(n-1)\Big(\tfrac{1}{2}\Big)^{n-1} \Gamma^c{}_{ba_2}\, h^{aa_2\ldots a_n} \dot{x}_c \dot{x}_{a_3} \cdots \dot{x}_{a_n} \Bigg]$$
$$+ \tfrac{1}{2} \dot{x}^c \dot{x}^d \Gamma_{dbc} \Big[n(n-1)\Big(\tfrac{1}{2}\Big)^{n-2} h^{aba_3\ldots a_n} \dot{x}_{a_3} \cdots \dot{x}_{a_n} \Big] + \dot{x}^b \Gamma^a{}_{bd} \Big[n\Big(\tfrac{1}{2}\Big)^{n-1} h^{da_2\ldots a_n} \dot{x}_{a_2} \cdots \dot{x}_{a_n} \Big]$$
$$= n\Big(\tfrac{1}{2}\Big)^{n-1} \dot{x}^b (\nabla_b h^{aa_2\ldots a_n}) \dot{x}_{a_2} \cdots \dot{x}_{a_n}. \tag{101}$$

Therefore, the geodesic equation becomes,

$$\ddot{x}^a + \Gamma^a{}_{bd} \dot{x}^d \dot{x}^b = \epsilon \Bigg[-\Big(\tfrac{1}{2}\Big)^{n-1} g^{ab}\, (\nabla_b h^{c_1\ldots c_n})\, \dot{x}_{c_1} \cdots \dot{x}_{c_n} \ + \ n\Big(\tfrac{1}{2}\Big)^{n-1} \dot{x}^b \nabla_b h^{ac_2\ldots c_n}\, \dot{x}_{c_2} \cdots \dot{x}_{c_n} \Bigg].$$
$$= \epsilon \Big(\tfrac{1}{2}\Big)^{n-1} g^{ab} \Big(n\, \nabla_{c_1} h_{(bc_2\ldots c_n)} - \nabla_b h_{(c_1\ldots c_n)} \Big)\, \dot{x}^{c_1} \cdots \dot{x}^{c_n}. \tag{102}$$

This compact formula for the n-th order polynomial perturbation is one of the central results of this thesis and it motivates a general conjecture about the structure of modified geodesic equations. It illustrates why an antisymmetric structure always shows up in the geodesic equation for any first-order perturbation: the term naturally occurs for perturbations homogeneous in momenta regardless of it's structure. We also see that, the velocity dependence on the right-hand side can be completely factorized, so that only covariant derivatives of the perturbation tensor $h^{c_1 c_2 \cdots c_n}$ remain. This means that the resulting geodesic equation retains a fully tensorial form on the base manifold. This formula's power is in its scope. By simply replacing the explicit form of the polynomial into this expression, we may obtain the modified geodesic equation for any polynomial perturbation $h(x,p)$. This enables a broad range of models to be handled within a single, cohesive framework, such as those derived from string theory or other quantum-gravity scenarios. However, the formula is restricted to perturbations that are polynomial and homogeneous in the momenta; more exotic modifications that are non-polynomial, non-homogeneous, or involve nonlocal structures fall outside its reach. Having established this general Hamiltonian result, we now turn to the Lagrangian picture, where it is instructive to see how these structures reappear when recast in velocity space.

VI. GEOMETRY OF THE TANGENT BUNDLE: FINSLER GEOMETRY

There are complementary reasons to recast the problem in velocity space. Many physical observables and comparisons with general relativity are most transparent in a Lagrangian (tangent-bundle) language where one varies a length functional along curves $x(\tau)$. A route that we could follow is to take $H(x,p)$, perform a Legendre transform to obtain $L(x,\dot{x})$, and then vary an action built from L. In this section, I give a short overview of how the Lagrangian is constructed and how the geodesic equation can be derived from it. Essentially, the goal is just to briefly introduce Finsler geometry, summarize the procedure and highlight the connection between the Lagrangian setup and the resulting geodesic dynamics.

The idea that would later develop into Finsler geometry goes back to Bernhard Riemann's 1854 Habilitation Dissertation. In the second part of that work, Riemann suggested that instead of measuring distances only with the usual quadratic form ($ds = \sqrt{g_{ij}dx^i dx^j}$), one could also imagine more general ways of defining the line element. For example, he proposed that the line element could be the fourth root of a quartic differential expression. This opened up the possibility of line elements beyond the standard Pythagorean type, where the metric structure is quadratic. Riemann restricted his own study to the quadratic case, but he pointed out that more general cases would take longer to develop and would be harder to express with clear results.

It was only much later, in 1918, that Paul Finsler in his PhD thesis took up this more general idea. Finsler defined distances using a function that is homogeneous of degree one in the tangent directions. In other words, in Finsler geometry, the distance between points depends not only on the points themselves but also on the directions in which one moves. This dependence on direction is exactly what marks the non-Pythagorean nature of such spaces.

Later developments connected this idea to physics. In particular, when considering modified dispersion relations in the dynamics of particles, the usual quadratic relation is no longer valid, and one is naturally led to consider Finsler-type structures. The modified dispersion relations can be described by a Hamiltonian constraint. Free particles, whose motion follows from such Hamiltonians, can be described by trajectories obtained from a Helmholtz-type action principle. For a particle, this action is given by

$$S[x,p,\lambda] = \int d\mu, (\dot{x}^a p_a - \lambda(H(x,p) - m)), \tag{103}$$

where p_a is the momentum, $H(x,p)$ is the Hamiltonian, m is the mass, and λ is a Lagrange multiplier enforcing the mass-shell condition.

By varying this action, one recovers the dispersion relation and the equations of motion. For example, let us consider the general first-order perturbation to the dispersion relation:

$$H(x,p) = g(p,p) + \epsilon h(x,p), \tag{104}$$

To pass from the dispersion relation to a length measure along trajectories, use the Helmholtz action

$$S[x,p,\lambda]_H = \int d\mu \left(\dot{x}^a p_a - \lambda f(H(x,p), m)\right), \tag{105}$$

where λ enforces the mass shell and $f = 0$ is equivalent to $H = m^2$. The algorithm (applied perturbatively in ϵ) is:

- Varying with respect to λ imposes the dispersion relation, i.e. $\delta_\lambda S \Rightarrow f = 0 \Rightarrow H(x,p) = m^2$.

- Varying with respect to the momentum p_μ gives a relation between velocity and momentum i.e. $\delta_{p_a} S \Rightarrow \dot{x}^a = \lambda\, \partial_{\bar{a}} H\, \partial_H f$. With $H = g + \epsilon h$,

$$\dot{x}^a = \lambda\,(2p^a + \epsilon\,\partial_{\bar{a}} h)\,\partial_H f \quad \Longrightarrow \quad p_a = \frac{\dot{x}_a}{2\lambda\,\partial_H f} - \frac{\epsilon}{2}\, g_{ab}\partial_{\bar{b}} h. \tag{106}$$

  From this follow the useful identities

$$\dot{x}^a p_a = \lambda\,(2g + \epsilon\, p\cdot\partial h)\,\partial_H f, \qquad g(p,p) = \frac{g(\dot{x},\dot{x})}{4\lambda^2(\partial_H f)^2} - \epsilon\,\frac{\dot{x}^a \partial_{\bar{a}} h}{2\lambda\,\partial_H f}. \tag{107}$$

- Solving this relation allows us to eliminate the momentum in terms of velocity. Therefore, Insert the last relation into $H = m^2$ and solve order by order with $\lambda = \lambda_0 + \epsilon\lambda_1$:

$$\lambda_0 = \frac{\sqrt{g(\dot{x},\dot{x})}}{2m\,\partial_H f}, \qquad \lambda_1 = \frac{\sqrt{g(\dot{x},\dot{x})}}{4m^3\,\partial_H f}\left(h - \frac{\dot{x}^a \partial_{\bar{a}} h}{2m}\right). \tag{108}$$

- Substitute $p(x, \dot{x}, \lambda)$ and $\lambda(x, \dot{x})$ back into $S[x, p, \lambda]_H$. All appearances of f cancel at first order and one obtains the Finsler length $S[x] = \int F(x, \dot{x})\, d\tau$ with

$$F(x, \dot{x}) = m\sqrt{g(\dot{x}, \dot{x})} - \epsilon \frac{\sqrt{g(\dot{x}, \dot{x})}}{2m}\, h\big(x, \bar{p}(x, \dot{x})\big), \quad \bar{p}_a := m\,\frac{g_{ab}\dot{x}^b}{\sqrt{g(\dot{x}, \dot{x})}} \quad \text{(Eq. (10))}. \tag{109}$$

This is the Finsler function induced by a general first-order MDR; it depends on x and $\dot{x}$ and is 1-homogeneous in $\dot{x}$. This procedure is essentially a Legendre transformation, which converts the Hamiltonian formulation into a Lagrangian one. The effective length measure defines an arc-length functional that can be identified with the geometry of the underlying space.

This can be written in terms of a generalized arc-length, showing that the motion of a particle subject to a modified dispersion relation corresponds to extremizing a deformed length functional. Identifying this functional with an arc-length in a generalized geometry is what connects the mechanics of free particles with Finsler geometry.

Thus, the geometry in which particles propagate with MDRs is naturally described by a generalized arc-length functional of the form

$$s[x] = \int F(x, \dot{x}), d\mu, \tag{110}$$

where $F(x, \dot{x})$ is a positively homogeneous function of degree one in the velocity. This ensures parametrization invariance of the arc-length.

Finsler Spacetimes

To formalize this, one works with the tangent bundle TM, with coordinates (x, y) where x are spacetime coordinates and y are velocity coordinates. One typically considers the slit tangent bundle $TM \setminus 0$, excluding the zero vector.

A conic subbundle $\mathcal{D} \subset \widetilde{TM}$ is introduced, satisfying $\pi(\mathcal{D}) = M$ under the projection $\pi : TM \to M$, and with the property that if $(x, y) \in \mathcal{D}$, then also $(x, \lambda y) \in \mathcal{D}$ for all $\lambda > 0$. This ensures the cone-like structure in each fiber of the tangent bundle.

A Finsler spacetime is then defined as a triple $(M, \mathcal{D}, L)$, where $L : \mathcal{D} \to \mathbb{R}$ is a smooth function satisfying:

- Positive 2-homogeneity: $L(x, \alpha y) = \alpha^2 L(x,y)$ for all $\alpha > 0$.

- 2. Non-degenerate Hessian: At every $(x,y) \in \mathcal{D}$, the Hessian
$$g_{\mu\nu}(x,y) = \frac{1}{2}\frac{\partial^2 L(x,y)}{\partial y^\mu \partial y^\nu} \tag{111}$$
is non-degenerate.

- Lorentzian signature: The metric $g_{\mu\nu}$ has Lorentzian signature.

The function L is essentially the square of the Finsler function, $L(x,y) = F^2(x,y)$. Using Euler's Theorem for homogeneous Function, the Finsler arc-length is defined as

$$s[x] = \int \sqrt{g_{\mu\nu}(x,\dot{x}), \dot{x}^\mu \dot{x}^\nu}, d\mu. \tag{112}$$

Because of the homogeneity condition, this definition does not depend on the choice of parametrization.

The Finsler metric $g_{\mu\nu}(x,y)$ transforms under coordinate changes in a way similar to the Riemannian case but with dependence on both position and direction. This makes it a distinguished tensor field on the tangent bundle.

Extremizing the Finsler arc-length yields the second-order ODE
$$\frac{d^2 x^\mu}{d\mu^2} + 2G^\mu(x,\dot{x}) = 2\frac{dF}{d\mu}\frac{\partial F}{\partial \dot{x}^\mu}. \tag{1}$$
The functions $G^\mu(x,\dot{x})$ are the spray coefficients of the Finsler structure. They are quadratic in the velocities and can be written as
$$G^\alpha(x,\dot{x}) = \frac{1}{2}\hat{\gamma}^{\alpha\mu\nu}(x,\dot{x}), \dot{x}^\mu \dot{x}^\nu, \tag{113}$$
where $\hat{\gamma}^{\alpha\mu\nu}(x,\dot{x})$ are the Christoffel-type symbols built from the Finsler metric $g_{\mu\nu}(x,\dot{x}) = \frac{1}{2}\frac{\partial^2 L}{\partial \dot{x}^\mu \partial \dot{x}^\nu}$ via
$$\hat{\gamma}^{\alpha\mu\nu}(x,\dot{x}) = \frac{1}{2}g^{\alpha\beta}(x,\dot{x})\left(\frac{\partial g^{\mu\beta}}{\partial x^\nu} + \frac{\partial g_{\nu\beta}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\beta}\right). \tag{114}$$
where the spray coefficients $G^\mu(x,\dot{x})$ encode the effect of the Finsler structure. These coefficients are built from the Finsler metric and play the role of generalized Christoffel symbols.

In this way, Finsler geometry generalizes Riemannian geometry by introducing direction dependence into the metric structure, which naturally arises when considering MDRs and quantum gravity motivated deformations.

VII. DERIVING GEODESIC EQUATION FROM THE FINSLER FUNCTION

In this section we turn to the Lagrangian formulation of MDRs and derive the geodesic equation from the Finsler function. The goal here is to establish a fully general expression for particle trajectories in the Lagrangian picture, which complements the Hamiltonian framework developed earlier. This result is important because it provides a second, independent route to the geodesic equation and allows us to compare systematically the two approaches. Having this parallel formulation will also be essential later, when we revisit explicit examples, in particular when we derive the geodesic equation for an MDR with with a n-th order polynomial term in p, and examine how the structure of the geodesic equation changes depending on the structure of the perturbation function. With the Finsler function $F(x,\dot{x})$ at hand, one can proceed to derive the geodesic equation directly from the Euler–Lagrange equations,

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^a}\right) - \frac{\partial L}{\partial x^a} = 0, \tag{115}$$

with $L = F^2$. In this framework the dynamics takes the form of a Finsler geodesic equation,

$$\ddot{x}^a + 2G^a(x,\dot{x}) = 0, \tag{116}$$

where the geodesic spray G^a is expressed in terms of the inverse Finsler metric g_L^{ab} and the derivatives of L as

$$G^a = \tfrac{1}{4}\, g_L^{ab}\big(\dot{x}^c \partial_c \dot{\partial}_b L - \partial_b L\big). \tag{117}$$

Hence, we should be able to derive the geodesic equation if we are able to derive the inverse Finsler metric g^{Lab}, which is defined by

$$g^{Lab} g^L_{bc} = \delta^a_c \tag{118}$$

and

$$g^L_{bc} = \frac{1}{2}\dot{\partial}_b \dot{\partial}_c L\,. \tag{119}$$

For

$$L = F^2 = g(\dot{x},\dot{x})\left(m^2 - \epsilon h(x,\dot{x})\right) . \tag{120}$$

we find

$$g^L_{ab} = m^2 g_{ab} - \epsilon\left(g_{ab}h + \dot{x}_a\dot{\partial}_b h + \dot{x}_b\dot{\partial}_a h + \frac{1}{2}g(\dot{x},\dot{x})\dot{\partial}_a\dot{\partial}_b h\right) \tag{121}$$

with inverse

$$g^{Lab} = \frac{1}{m^2}g^{ab} + \frac{\epsilon}{m^4}\left(g^{ab}h + (\dot{x}^a g^{bc} + \dot{x}^b g^{ac})\dot{\partial}_c h + \frac{1}{2}g(\dot{x},\dot{x})g^{ac}g^{bd}\dot{\partial}_c\dot{\partial}_d h\right) . \tag{122}$$

Now, the derivatives of L are

$$\begin{aligned} \partial_a L &= 2\Gamma_{iaj}\dot{x}^i\dot{x}^j(m^2 - \epsilon h) - \epsilon g(\dot{x},\dot{x})\partial_a h\,, \\ \dot{\partial}_a L &= 2\dot{x}_a(m^2 - \epsilon h) - \epsilon g(\dot{x},\dot{x})\dot{\partial}_a h\,, \\ \dot{\partial}_b\partial_a L &= 2(\Gamma_{iab} + \Gamma_{bai})\dot{x}^i(m^2 - \epsilon h) - \epsilon\left(g(\dot{x},\dot{x})\dot{\partial}_b\partial_a h + 2(\Gamma_{iaj}\dot{x}^i\dot{x}^j\dot{\partial}_b h + \dot{x}_b\partial_a h)\right) . \end{aligned} \tag{123}$$

Thus we find

$$\begin{aligned} &\dot{x}^a\partial_a\dot{\partial}_b L - \partial_b L \\ &= 2m^2\Gamma_{bij}\dot{x}^i\dot{x}^j - \epsilon\left(g(\dot{x},\dot{x})(\dot{x}^a\partial_a\dot{\partial}_b h - \partial_b h) + 2(\dot{x}_b\dot{x}^a\partial_a h + \dot{\partial}_b h\Gamma_{ijk}\dot{x}^i\dot{x}^j\dot{x}^k + h\Gamma_{bij}\dot{x}^i\dot{x}^j)\right) . \end{aligned} \tag{124}$$

This term can be mulitplied with the inverse Finsler metric

$$\begin{aligned} &g^{Lqb}(\dot{x}^a\partial_a\dot{\partial}_b L - \partial_b L) \\ &= \left(\frac{1}{m^2}g^{qb} + \frac{\epsilon}{m^4}\left(g^{qb}h + (\dot{x}^q g^{bc} + \dot{x}^b g^{qc})\dot{\partial}_c h + \frac{1}{2}g(\dot{x},\dot{x})g^{qc}g^{bd}\dot{\partial}_c\dot{\partial}_d h\right)\right) \\ &\times\left(2m^2\Gamma_{bij}\dot{x}^i\dot{x}^j - \epsilon\left(g(\dot{x},\dot{x})(\dot{x}^a\partial_a\dot{\partial}_b h - \partial_b h) + 2(\dot{x}_b\dot{x}^a\partial_a h + \dot{\partial}_b h\Gamma_{ijk}\dot{x}^i\dot{x}^j\dot{x}^k + h\Gamma_{bij}\dot{x}^i\dot{x}^j)\right)\right) \\ &= 2\Gamma^q{}_{ij}\dot{x}^i\dot{x}^j - \frac{\epsilon}{m^2}\left(g(\dot{x},\dot{x})g^{qb}(\dot{x}^a\partial_a\dot{\partial}_b h - \partial_b h - \Gamma^d{}_{ij}\dot{x}^i\dot{x}^j\dot{\partial}_d\dot{\partial}_b h) + 2\dot{x}^q\dot{x}^a(\partial_a h - \Gamma^c{}_{aj}\dot{x}^j\dot{\partial}_c h)\right) . \end{aligned} \tag{125}$$

Hence, the geodesic spray coefficients G^q are

$$\begin{aligned} 2G^q &= \frac{1}{2}\left(g^{Lqb}(\dot{x}^a\partial_a\dot{\partial}_b L - \partial_b L)\right) \\ &= \Gamma^q{}_{ij}\dot{x}^i\dot{x}^j - \frac{\epsilon}{2m^2}\left(g(\dot{x},\dot{x})g^{qb}(\dot{x}^a\partial_a\dot{\partial}_b h - \partial_b h - \Gamma^d{}_{ij}\dot{x}^i\dot{x}^j\dot{\partial}_d\dot{\partial}_b h) + 2\dot{x}^q\dot{x}^a(\partial_a h - \Gamma^c{}_{aj}\dot{x}^j\dot{\partial}_c h)\right) , \end{aligned} \tag{126}$$

and thus the geodesic equation can be written as

$$\ddot{x}^q + \Gamma^q{}_{ij}\dot{x}^i\dot{x}^j = \frac{\epsilon}{2m^2}\left(g(\dot{x},\dot{x})g^{qb}(\dot{x}^a\partial_a\dot{\partial}_b h - \partial_b h - \Gamma^d{}_{ij}\dot{x}^i\dot{x}^j\dot{\partial}_d\dot{\partial}_b h) + 2\dot{x}^q\dot{x}^a(\partial_a h - \Gamma^c{}_{aj}\dot{x}^j\dot{\partial}_c h)\right) . \tag{127}$$

This is the desired geodesic equation.

In Finsler Language, using the horizontal derivative, $\delta_a = \partial_a - \Gamma^c{}_{ab}\dot{x}^b\dot{\partial}_c$, it can be rewritten as

$$\ddot{x}^q + \Gamma^q{}_{ij}\dot{x}^i\dot{x}^j = \frac{\epsilon}{2m^2}\left(g(\dot{x},\dot{x})g^{qb}(\dot{x}^a\delta_a\dot{\partial}_b h - \Gamma^c{}_{bj}\dot{x}^j\dot{\partial}_c h - \delta_b h) + 2\dot{x}^q\dot{x}^a\delta_a h\right) \tag{128}$$

This compact expression provides another central finding of this thesis. It is the Lagrangian analogue of the Hamiltonian first-order result (77). In order to see explicitly how the two formulations compare, it is useful to first examine the simplest case of a linear perturbation.

In general we know from [20] that, coming from a modified dispersion relation, h needs to take the form

$$h(x,\dot{x}) = h(x,\bar{p}(x,\dot{x}))\,, \quad \bar{p}_a = m\frac{g_{ab}(x)\dot{x}^b}{\sqrt{g_{ij}(x)\dot{x}^i\dot{x}^j}} = m\dot{\partial}_a\sqrt{g_{ij}(x)\dot{x}^i\dot{x}^j} = m\dot{\partial}_a\ell\,. \tag{129}$$

The last way of writing the momenta is very convenient when we derive the geodesic equation, since

$$\delta_a\ell = 0\,, \tag{130}$$

$$\delta_b\bar{p}_a = m\delta_b\dot{\partial}_a\ell = m\delta_b\frac{g_{ac}\dot{x}^c}{\ell} = \frac{m}{\ell}\delta_b(g_{ac}\dot{x}^c) = \frac{m}{\ell}\dot{x}^c\Gamma_{cba} = \bar{p}_c\Gamma^c{}_{ba}\,, \tag{131}$$

$$\dot{\partial}_b\bar{p}_a = m\dot{\partial}_b\dot{\partial}_a\ell = \frac{m}{\sqrt{g(\dot{x},\dot{x})}}\left(g_{ab} - \frac{\dot{x}_a\dot{x}_b}{g(\dot{x},\dot{x})}\right) \tag{132}$$

Therefore, a modified dispersion relation with linear term in p

$$h(x,p) = A^a(x)p_a \Rightarrow h(x,\dot{x}) = h(x,\bar{p}(x,\dot{x})) = mA^a(x)\frac{g_{ab}(x)\dot{x}^b}{\sqrt{g_{ij}(x)\dot{x}^i\dot{x}^j}} = m\frac{g(A,\dot{x})}{\sqrt{g(\dot{x},\dot{x})}}\,. \tag{133}$$

Alternatively we could also write

$$h(x,\dot{x}) = h(x,\bar{p}(x,\dot{x})) = mA^a(x)\dot{\partial}_a\ell(x,\dot{x})\,. \tag{134}$$

For the geodesic equation we need some derivatives of h

$$\delta_a h = (\delta_a A^c)\bar{p}_c + A^c \delta_a \bar{p}_c = \partial_a A^c \bar{p}_c + A^c \Gamma^q{}_{ac}\bar{p}_q = \bar{p}_c \nabla_a A^c \tag{135}$$

$$\dot{\partial}_a h = A^c \dot{\partial}_a \bar{p}_c = \frac{m}{\sqrt{g(\dot{x},\dot{x})}} A^c \left(g_{ac} - \frac{\dot{x}_a \dot{x}_c}{g(\dot{x},\dot{x})}\right) = \frac{m}{\ell} A^c \left(g_{ac} - \frac{\bar{p}_a \bar{p}_c}{m^2}\right) \tag{136}$$

$$\begin{aligned} \delta_b \dot{\partial}_a h &= \frac{m}{\ell}\delta_b A^c \left(g_{ac} - \frac{\bar{p}_a\bar{p}_c}{m^2}\right) + \frac{m}{\ell} A^c \left(\delta_b g_{ac} - \frac{\delta_b \bar{p}_a \bar{p}_c + \bar{p}_a \delta_b \bar{p}_c}{m^2}\right) \\ &= \frac{m}{\ell}\partial_b A^c \left(g_{ac} - \frac{\bar{p}_a\bar{p}_c}{m^2}\right) + \frac{m}{\ell} A^c \left(\Gamma_{abc} + \Gamma_{cba} - \frac{\bar{p}_d \Gamma^d{}_{ab}\bar{p}_c + \bar{p}_a \bar{p}_d \Gamma^d{}_{cb}}{m^2}\right). \end{aligned} \tag{137}$$

These terms can be combined to

$$\begin{aligned} &\dot{x}^a \delta_a \dot{\partial}_b h - \Gamma^c{}_{bj}\dot{x}^j \dot{\partial}_c h \\ &= \frac{m}{\ell}\left(\dot{x}^a \partial_a A^c \left(g_{bc} - \frac{\bar{p}_b \bar{p}_c}{m^2}\right) + A^c \dot{x}^a \left(\Gamma_{bac} + \Gamma_{cab} - \frac{\bar{p}_d \Gamma^d{}_{ab}\bar{p}_c + \bar{p}_b \bar{p}_d \Gamma^d{}_{ca}}{m^2}\right) - \Gamma^c{}_{bj}\dot{x}^j A^d \left(g_{cd} - \frac{\bar{p}_c \bar{p}_d}{m^2}\right)\right) \\ &= \frac{m}{\ell}\left(\dot{x}^a \partial_a A^c \left(g_{bc} - \frac{\dot{x}_b \dot{x}_c}{\ell^2}\right) + A^c \dot{x}^a \left(\Gamma_{bac} - \frac{\bar{p}_b \bar{p}_d \Gamma^d{}_{ca}}{m^2}\right)\right) \\ &= \frac{m}{\ell}\left(g_{bd}\dot{x}^a (\partial_a A^d + A^c \Gamma^d{}_{ac}) - \frac{\dot{x}^a \bar{p}_b}{m^2}\bar{p}_d (\partial_a A^d + A^c \Gamma^d{}_{ca})\right) \\ &= \frac{m}{\ell}\left(g_{bd} - \frac{\bar{p}_b \bar{p}_d}{m^2}\right) \dot{x}^a \nabla_a A^d . \end{aligned} \tag{138}$$

Hence the whole geodesic equation (128) becomes

$$\begin{aligned} \ddot{x}^q + \Gamma^q{}_{ij}\dot{x}^i \dot{x}^j &= \frac{\epsilon}{2m^2}\left(g(\dot{x},\dot{x}) g^{qb} \frac{m}{\ell}\left(g_{bd} - \frac{\bar{p}_b \bar{p}_d}{m^2}\right) \dot{x}^a \nabla_a A^d - g(\dot{x},\dot{x}) g^{qb}\bar{p}_c \nabla_b A^c + 2\dot{x}^q \dot{x}^a \bar{p}_c \nabla_a A^c \right) \\ &= \frac{\epsilon}{2m^2}\left(g(\dot{x},\dot{x}) \frac{m}{\ell}\left(\delta^q_d - \frac{\bar{p}^q \bar{p}_d}{m^2}\right) \dot{x}^a \nabla_a A^d - g(\dot{x},\dot{x}) g^{qb}\bar{p}_c \nabla_b A^c + 2\dot{x}^q \dot{x}^a \bar{p}_c \nabla_a A^c \right) \\ &= \frac{\epsilon}{2m^2}\left(\ell m \left(\delta^q_d - \frac{\dot{x}^q \dot{x}_d}{\ell^2}\right) \dot{x}^a \nabla_a A^d - m\ell g^{qb}\dot{x}_c \nabla_b A^c + 2\frac{m}{\ell}\dot{x}^q \dot{x}^a \dot{x}_c \nabla_a A^c \right) \\ &= \frac{\epsilon}{2m}\left(\ell \left(\dot{x}^a \nabla_a A^q - g^{qb}\dot{x}_c \nabla_b A^c \right) + \frac{1}{\ell}\dot{x}^q \dot{x}^a \dot{x}_c \nabla_a A^c \right) \\ &= \frac{\epsilon}{2m}\left(\ell \dot{x}^a g^{qb} \left(\nabla_a A_b - \nabla_b A_c \right) + \frac{\dot{x}^q}{\ell}\dot{x}^a \dot{x}^c \nabla_a A_c \right). \end{aligned} \tag{139}$$

On first sight, this equation looks different compared to (79). However we can change the parametrization $\tau \to s(\tau)$ so that

$$\begin{aligned} \dot{x}(s(\tau)) &= \frac{d}{d\tau} x(s(\tau)) = \frac{dx}{ds}\frac{ds}{d\tau} = x' \dot{s} , \\ \ddot{x}(s(\tau)) &= x' \ddot{s} + x'' \dot{s}^2 , \end{aligned} \tag{140}$$

which turns the geodesic equation (139) to

$$x''^q + \Gamma^q{}_{bc}x'^b x'^c = \frac{\epsilon}{m}\left(\ell(x')x'^a g^{qb}\left(\nabla_a A_b - \nabla_b A_c\right)\right) + x'^q\left(\frac{\epsilon}{m\ell(x')}x'^c x'^d \nabla_c A_d - \frac{\ddot{s}}{\dot{s}^2}\right). \tag{141}$$

The last term vanishes if one choose a parametrization of the curve such that

$$\frac{\ddot{s}}{\dot{s}^2} = \frac{\epsilon}{m\ell(x')}x'^c x'^d \nabla_c A_d\,, \tag{142}$$

which that turns the geodesic equation closer to (79), which was given by

$$\ddot{x}^q + \Gamma^q{}_{ij}\dot{x}^i\dot{x}^j = \epsilon\dot{x}^p g^{ab}\left[\nabla_p A_a - \nabla_a A_p\right]. \tag{143}$$

We see that geodesic equation of GR is reobtained in both cases if

$$\dot{x}^p g^{ab}\left[\nabla_p A_a - \nabla_a A_p\right] = 0\,. \tag{144}$$

In particular, if $A_a = \partial_a\phi$, the antisymmetric term vanishes identically and the GR geodesic equation is fully recovered, consistent with our discussion in the Hamiltonian formulation.

With the geodesic equation (128) in hand, we can practically derive all of the cases we previously covered in the Hamiltonian framework. I have worked out the two cubic cases: $h = g(p,p)A(p)$ and $A(p)^3$ and the more general case $g(p,p)^r A(p)^s$. In all th cases, the resulting geodesic equations are quite similar to those derived in the Hamiltonian picture, with the exception of an additional term that can be absorbed by curve reparametrization. After performing these computations, I explicitly determined which reparametrizations are required to achieve exact agreement with the Hamiltonian model. The computation details are available in Appendix B.

A. For modified dispersion relations with a n-th order polynomial term in p

In this section, we take the example of when the perturbation function is taken to be an arbitrary n-th order polynomial in the momenta. This example has already been worked out in the Hamiltonian formalism, and to repeat it here in the Lagrangian framework is particularly interesting. The calculation is noteworthy because it presents with a broad conclusion: the Lagrangian and Hamiltonian pictures yield equivalent dynamics, with their only difference being an additional term which can always be removed by reparametrization of the curve.

In this sense, the distinction between the two formulations reflects a freedom of parametrization rather than a physical discrepancy. At the same time, the antisymmetric structure that appears in the geodesic equation is not tied to the framework but is a characteristic feature of first-order polynomial perturbations that are homogeneous in the momenta.

Now, we start with our derivation where the perturbation function

$$h(x,p) = h^{a_1...a_n}(x)p_{a_1}...p_{a_n} \Rightarrow h(x,\dot{x}) = h(x,\bar{p}(x,\dot{x})) = m^n h^{a_1...a_n}(x)\frac{\dot{x}_{a_1}...\dot{x}_{a_n}}{\sqrt{g(\dot{x},\dot{x})}^n}\,. \tag{145}$$

For the geodesic equation we need some derivatives of h

$$\begin{aligned}
\delta_a h &= (\nabla_a h^{b_1...b_n})\bar{p}_{b_1}...\bar{p}_{b_n} \\
\dot{\partial}_a h &= n h^{(b_1 b_2...b_n)}\bar{p}_2...\bar{p}_n\dot{\partial}_a\bar{p}_{b1} = \frac{m\ n}{\sqrt{g(\dot{x},\dot{x})}}h^{(b_1 b_2...b_n)}\bar{p}_2...\bar{p}_n\left(g_{ab_1} - \frac{\dot{x}_a\dot{x}_{b_1}}{g(\dot{x},\dot{x})}\right) \\
\delta_c\dot{\partial}_a h &= \frac{m\ n}{\sqrt{g(\dot{x},\dot{x})}}\Bigg(\left((\delta_c h^{(b_1 b_2...b_n)})\bar{p}_2...\bar{p}_n + (n-1)h^{(b_1 b_2...b_n)}\bar{p}_2...\delta_c\bar{p}_n\right)\left(g_{ab_1} - \frac{\dot{x}_a\dot{x}_{b_1}}{g(\dot{x},\dot{x})}\right) \\
&\quad + h^{(b_1 b_2...b_n)}\bar{p}_2...\bar{p}_n\delta_c\left(g_{ab_1} - \frac{\dot{x}_a\dot{x}_{b_1}}{g(\dot{x},\dot{x})}\right)\Bigg)
\end{aligned} \tag{146}$$

Evaluating the following expression:

$$
\begin{aligned}
&\dot{x}^c\delta_c\dot{\partial}_a h - \Gamma^c{}_{aj}\dot{x}^j\dot{\partial}_c h \\
&= \frac{m\ n}{\sqrt{g(\dot{x},\dot{x})}}\Bigg[h^{(b_1b_2...b_n)}\bar{p}_2...\bar{p}_{b_n}\ \dot{x}^c\Gamma^r{}_{cb_1}\left(g_{ra} - \frac{\dot{x}_a\dot{x}_r}{g(\dot{x},\dot{x})}\right) \\
&+\dot{x}^c\left((\partial_c h^{(b_1b_2...b_n)})\bar{p}_2...\bar{p}_{b_n} + (n-1)h^{(b_1b_2...b_n)}\bar{p}_2...\bar{p}_r\Gamma^r{}_{cb_n}\right)\left(g_{ab_1} - \frac{\dot{x}_a\dot{x}_{b_1}}{g(\dot{x},\dot{x})}\right)\Bigg] \\
&= \frac{m\ n}{\sqrt{g(\dot{x},\dot{x})}}\Bigg[h^{(b_1b_2...b_n)}\bar{p}_2...\bar{p}_{b_n}\ \dot{x}^c\Gamma^r{}_{cb_1}\left(g_{ra} - \frac{\dot{x}_a\dot{x}_r}{g(\dot{x},\dot{x})}\right) \\
&+\dot{x}^c\left((\partial_c h^{(b_1b_2...b_n)})\bar{p}_2...\bar{p}_{b_n} + (n-1)h^{(b_1b_2...r)}\bar{p}_2...\bar{p}_{b_n}\Gamma^{b_n}{}_{cr}\right)\left(g_{ab_1} - \frac{\dot{x}_a\dot{x}_{b_1}}{g(\dot{x},\dot{x})}\right)\Bigg] \\
&= \frac{m\ n}{\sqrt{g(\dot{x},\dot{x})}}\Bigg[h^{(rb_2..b_{n-1}b_n)}\Gamma^d{}_{cr}\left(g_{da} - \frac{\dot{x}_a\dot{x}_d}{g(\dot{x},\dot{x})}\right) \\
&+\left((\partial_c h^{(b_1b_2...b_n)}) + (n-1)h^{(b_1b_2..b_{n-1}r)}\Gamma^{b_n}{}_{cr}\right)\left(g_{ab_1} - \frac{\dot{x}_a\dot{x}_{b_1}}{g(\dot{x},\dot{x})}\right)\Bigg]\bar{p}_2...\bar{p}_{b_n}\dot{x}^c \\
&= \frac{m\ n}{\sqrt{g(\dot{x},\dot{x})}}\Bigg[h^{(rb_2..b_{n-1}b_n)}\Gamma^{b_1}{}_{cr}\left(g_{b_1a} - \frac{\dot{x}_a\dot{x}_{b_1}}{g(\dot{x},\dot{x})}\right) \\
&+\left((\partial_c h^{(b_1b_2...b_n)}) + (n-1)h^{(b_1b_2..b_{n-1}r)}\Gamma^{b_n}{}_{cr}\right)\left(g_{ab_1} - \frac{\dot{x}_a\dot{x}_{b_1}}{g(\dot{x},\dot{x})}\right)\Bigg]\bar{p}_2...\bar{p}_{b_n}\dot{x}^c \\
&= \frac{m\ n}{\sqrt{g(\dot{x},\dot{x})}}\Bigg[h^{(rb_2..b_{n-1}b_n)}\Gamma^{b_1}{}_{cr} + \partial_c h^{(b_1b_2...b_n)} + (n-1)h^{(b_1b_2..b_{n-1}r)}\Gamma^{b_n}{}_{cr}\Bigg]\bar{p}_2...\bar{p}_{b_n}\dot{x}^c\left(g_{ab_1} - \frac{\dot{x}_a\dot{x}_{b_1}}{g(\dot{x},\dot{x})}\right) \\
&= \frac{m\ n}{\sqrt{g(\dot{x},\dot{x})}}\dot{x}^c\nabla_c h^{(b_1...b_n)}\bar{p}_2...\bar{p}_{b_n}\left(g_{ab_1} - \frac{\dot{x}_a\dot{x}_{b_1}}{g(\dot{x},\dot{x})}\right).
\end{aligned}
$$

Then, the full geodesic equation becomes

$$
\begin{aligned}
&\ddot{x}^q + \Gamma^q{}_{ij}\dot{x}^i\dot{x}^j \\
&= \frac{\epsilon}{2m^2}\Bigg(\ell^2 g^{qa}\left(\frac{m\ n}{\ell}\dot{x}^c\nabla_c h^{(b_1...b_n)}\bar{p}_2...\bar{p}_{b_n}\left(g_{ab_1} - \frac{\dot{x}_a\dot{x}_{b_1}}{\ell^2}\right) - (\nabla_a h^{b_1...b_n})\bar{p}_{b_1}...\bar{p}_{b_n}\right) \\
&\qquad + 2\dot{x}^q\dot{x}^a(\nabla_a h^{b_1...b_n})\bar{p}_{b_1}...\bar{p}_{b_n} \\
&= \frac{\epsilon}{2m^2}\Big(n\dot{x}^c\nabla_c h^{(b_1...b_n)}\bar{p}_2...\bar{p}_{b_n}\left(m\ell\delta^q_{b_1} - \bar{p}_{b_1}\dot{x}^q\right) - \ell^2 g^{qa}(\nabla_a h^{b_1...b_n})\bar{p}_{b_1}...\bar{p}_{b_n} \\
&\qquad + 2\dot{x}^q\dot{x}^a(\nabla_a h^{b_1...b_n})\bar{p}_{b_1}...\bar{p}_{b_n} \\
&= \frac{\epsilon}{2m^2}\left(n\ell^2\bar{p}^{b_1}\nabla_{b_1}h^{(q...b_n)}\bar{p}_2...\bar{p}_{b_n} - \ell^2 g^{qa}(\nabla_a h^{b_1...b_n})\bar{p}_{b_1}...\bar{p}_{b_n} + (2-n)\dot{x}^q\dot{x}^a(\nabla_a h^{b_1...b_n})\bar{p}_{b_1}...\bar{p}_{b_n}\right) \\
&= \frac{\epsilon}{2m^2}\left(\ell^2 g^{qa}\left(n\nabla_{b_1}h_{(a...b_n)} - \nabla_a h_{(b_1...b_n)}\right) + (2-n)\dot{x}^q\dot{x}^a(\nabla_a h_{b_1...b_n})\right)\bar{p}^{b_1}...\bar{p}^{b_n}. \qquad (147)
\end{aligned}
$$

So we can see that, in the Lagrangian picture, the derivation for the n-th order polynomial perturbation leads to almost the same compact structure as in the Hamiltonian case (102) except for the last term. The antisymmetric term again appears universally, confirming that this feature is not tied to the specific structure of the perturbation function or the framework, but is a general consequence of first-order homogeneous perturbations in momenta. And once again, all velocity terms can be extracted from the right-hand side, leaving only covariant derivatives of the corresponding $h^{b_1 b_2 \ldots b_n}$. This mirrors the structure already observed in the Hamiltonian polynomial case, ensuring that the Lagrangian geodesic equation too remains purely tensorial on the base manifold. The only difference we can see between the geodesic equations is the presence of an additional contribution $(2-n)\dot{x}^q \dot{x}^a (\nabla_a h_{b_1 \ldots b_n})\bar{p}^{b_1} \ldots \bar{p}^{b_n}$ which is proportional to $\dot{x}^q$. This term is absent in the Hamiltonian formulation, but because it is aligned with the velocity it can always be absorbed by a reparametrization of the curve. Thus, the two approaches are dynamically equivalent, and any extra terms in the Lagrangian result reflect nothing more than gauge freedom.

By substituting the explicit form of polynomial $h(x,p)$ and the value of n into the n-polynomial formula (147), we can in principle obtain exactly the same geodesic equation that would result from a direct derivation in the Lagrangian picture using (128). In practice the n-polynomial formula can make the calculation much simpler, since instead of going through all the intermediate steps we only need to perform a more straightforward substitution at time. For this reason, the n-polynomial formula serves as a ready-to-use expression whenever one wants to test a specific perturbation function.

To check this, I first applied the formula to study the two cubic cases I studied earlier in Lagrangian framework using the geodesic equation (128). The detailed calculations can be found in B 4. In both the mixed cubic case $h = g(p,p)A(p)$, and the pure cubic case $h = A(p)^3$, , the two methods give exactly the same geodesic equation as (B18) and(B10). This provides a nice consistency check for this n-polynomial formula derivation.

I then turned to the string theory–inspired examples, such as $h = (A\cdot p)^n$ and $h = (g(p,p) - (A\cdot p)^2)^N$. For the first case, by substituting $h = (A\cdot p)^n$ into (147), we find geodesic equations that match those obtained in the Hamiltonian picture (A21), up to the usual reparametrization freedom. In particular, the A^q term disappears when $n = 1$, but for higher powers it persists, exactly as in the Hamiltonian case. The derivation can be found in Appendix B 4 c

In the next section, I will give the calculation and theoretical motivation for the perturbation $h = (g(p,p) - (A \cdot p)^2)^N$, as promised in the Hamiltonian picture discussion.

B. MDR for D-particle Backgrounds and Vacuum Refractive Indices

In this section, I will provide a brief theoretical motivation, following by the derivation of the geodesic equation for the perturbation function $h = (g(p,p) - (A \cdot p)^2)^N$. In the Hamiltonian framework, we discussed this case but derived it using the general geodesic equation rather than the nn-polynomial formula. Here, as previously indicated, we now carry out the derivation with the polynomial formula, which provides insight into how such calculations unfold for cases of this type.

In string/brane realizations of space-time foam, our 4-D world is a D3-brane moving through a bulk populated by D-particle defects. Neutral open-string states (photons and other Cartan excitations) can be temporarily captured by a D-particle and re-emitted; the lifetime of the intermediate string grows with the incident energy, $\Delta t \sim \alpha' E$ [21]. This leads to a subluminal, energy-dependent vacuum refractive index $n(E) \simeq 1 + E/M_{\text{QG}}$, consistent with time-lag data from MAGIC/HESS/Fermi and with the absence of birefringence; charged states (e.g. electrons) do not couple at leading order, thereby evading synchrotron bounds. Recoil of the D-particle during capture distorts the open-string metric and induces a momentum-dependent (Finsler-like) background and non-commutative structure—features that further modify dispersion but are suppressed by the (large) D-particle mass. Altogether, the foam acts as a Lorentz-violating medium with a preferred four-velocity A^a (the average recoil/rest frame of the defects), in which photon kinematics is controlled by the momentum component transverse to A^a. This motivates encoding the modification in scalars built from the projector orthogonal to A^a, e.g. $p_\perp^2 \propto g^{ab}p_a p_b - (A{\cdot}p)^2$, and, more generally, by polynomial deformations $\left(g^{ab}p_a p_b - (A{\cdot}p)^2\right)^N$. Such terms vanish on the GR light cone yet capture the stringy refractive effect when expanded to first order, while naturally accommodating the selection rules (Cartan states affected, no birefringence) and the Finsler/non-commutative imprints of D-particle recoil. These are exactly the phenomenological features extracted in the Type-IIB D-foam analysis: linear energy-dependent time delays for photons, suppressed recoil corrections, and a consistent mapping to gamma-ray constraints. Let

$$h(x,p) = \Big(g^{ab}p_a p_b - (A^q p_q)^2\Big)^N = \big(h^{ab}p_a p_b\big)^N, \qquad h^{ab} := g^{ab} - A^a A^b. \tag{148}$$

In this case, we need to insert $n = 2N$ in the polynomial formula, and $\nabla_c h^{ab} = -A^a \nabla_c A^b - A^b \nabla_c A^a$

Let $\chi(x,\bar{p}) := h^{ab}\bar{p}_a\bar{p}_b, \qquad \bar{p}^a = \frac{m}{\ell}\dot{x}^a, \quad \ell^2 = g(\dot{x},\dot{x}).$

$$\begin{aligned}
\nabla_a h_{b_1\ldots b_{2N}}\,\bar{p}^{b_1}\cdots\bar{p}^{b_{2N}} &= N\,\chi^{N-1}\,(\nabla_a h_{bc})\,\bar{p}^b\bar{p}^c,\\
&= -2N\,\chi^{N-1} A(\bar{p})(\bar{p}^c\nabla_a A_c)\\
&= -2N\,\bar{\chi}(\dot{x})^{N-1}\left(\tfrac{m}{\ell}A(\dot{x})\right)\tfrac{m}{\ell}\nabla_a A_c\dot{x}^c.
\end{aligned} \tag{149}$$

$$\begin{aligned}
\bar{p}^{b_1}\cdots\bar{p}^{b_{2N}}\;\bar{\nabla}_{b_1}h_{ab_2\cdots b_{2N}} &= \chi^{N-1}\,(\nabla_b h_{ac})\,\bar{p}^b\bar{p}^c \;+\; (N-1)\,\chi^{N-2}\,h_{ac}\bar{p}_c\,(\nabla_b h_{rs})\,\bar{p}^b\bar{p}^r\bar{p}^s\\
&= -\,\chi^{N-1}\Big[A_a(\bar{p}^b\nabla_b A_c\bar{p}^c) + A(\bar{p})(\bar{p}^b\nabla_b A_a)\Big]\\
&\qquad - 2(N-1)\chi^{N-2}A(\bar{p})(\bar{p}^b\nabla_b A_s\bar{p}^s)(g_{ac} - A_a A_c)\bar{p}^c.\\
&= -\,\bar{\chi}(\dot{x})^{N-1}\left[A_a\left(\tfrac{m}{\ell}\right)^2\dot{x}^b\nabla_b A_c\dot{x}^c + \left(\tfrac{m}{\ell}\right)^2 A(\dot{x})\left(\dot{x}^b\nabla_b A_a\right)\right]\\
&\qquad - 2(N-1)\,\bar{\chi}(\dot{x})^{N-2}\,A(\dot{x})\left(\tfrac{m}{\ell}\right)^4\dot{x}^b\nabla_b A_s\dot{x}^s\,\dot{x}_a\\
&\qquad + 2(N-1)\left(\tfrac{m}{\ell}\right)^4\bar{\chi}(\dot{x})^{N-2}\left(A(\dot{x})\right)^2\dot{x}^b\nabla_b A_s\dot{x}^s\,A_a.
\end{aligned} \tag{150}$$

Plugging these into

$$\text{RHS} = \frac{\epsilon}{2m^2}\Big(\ell^2 g^{qa}\Big((2N)\bar{\nabla}_{b_1}h_{(a\cdots)} - \nabla_a h_{(\ldots)}\Big) + (2-2N)\dot{x}^q\dot{x}^a\,\nabla_a h_{(\ldots)}\Big)\bar{p}^{b_1}\cdots\bar{p}^{b_{2N}}, \tag{151}$$

$$\begin{aligned}
\text{RHS} = \frac{\epsilon}{2m^2}\Bigg\{&2N\,m^2\,\chi^{N-1}\,\dot{x}^b\Big(A(\dot{x})\;\left(\nabla^q A_b - \nabla_b A^q\right) - A^q\nabla_b A_c\dot{x}^c\Big)\\
&+ 4N(N-1)\left(\tfrac{m}{\ell}\right)^2\bar{\chi}(\dot{x})^{N-2}\,A(\dot{x})\dot{x}^q\dot{x}^b\nabla_b A_s\dot{x}^s\Big(\bar{\chi}(\dot{x}) - m^2\Big)\\
&+ 4N(N-1)\,\tfrac{m^4}{\ell^2}\bar{\chi}(\dot{x})^{N-2}\left(A(\dot{x})\right)^2\dot{x}^b\nabla_b A_s\dot{x}^s\,A^q.
\end{aligned} \tag{152}$$

From Hamiltonian framework:

$$\ddot{x}^a + \Gamma^a{}_{bd}\dot{x}^d\dot{x}^b = \epsilon\Bigg[N\left(\tfrac{1}{4}(g(\dot{x},\dot{x}) - A(\dot{x})^2)\right)^{N-1}\left(A(\dot{x})\dot{x}^b(\nabla^a A_b - \nabla_b A^a) - A^a\dot{x}^b\dot{x}^m\nabla_b A_m\right) \\ + \tfrac{1}{2}N(N-1)\left(\tfrac{1}{4}g(\dot{x},\dot{x}) - \tfrac{1}{4}A(\dot{x})^2\right)^{N-2}\left(A(\dot{x})\,\dot{x}^b\,\dot{x}^m\,(-\dot{x}^a + A^a A(\dot{x}))\,\nabla_b A_m\right)\Bigg] \tag{153}$$

VIII. RESULTS AND DISCUSSION

This thesis set out to understand how modified dispersion relations (MDRs) affect the motion of freely falling particles. The study was carried out in two complementary pictures.

We first worked entirely within the Hamiltonian framework, where point-particle dispersion relations appear as level sets of Hamilton functions $H(x,p)$ defined on the cotangent bundle T^*M of spacetime. We began by deriving the geodesic equation for a general first-order perturbation to the relativistic Hamiltonian, $H(x,p) = g(p,p) + \epsilon\, h(x,p)$ in srction V. This expression (77) represents the most general result in the Hamiltonian picture. After obtaining this general form, we applied it to several explicit examples of perturbation functions to see how different structures emerge in practice. These examples, motivated by various theoretical models such as linear, cubic terms in momenta, helped to reveal which types of terms survive, cancel, or can be absorbed through reparametrization. Building on these observations, we then derived the homogeneous n-th-order polynomial formulation, which provides a compact and unified expression encompassing all polynomial perturbations as special cases.

In the second part, we reformulated the same problem in the Lagrangian picture using the Helmholtz action, which defines a Finsler-type "length" along worldlines. The Lagrangian is constructed from the corresponding Finsler function, and its Euler–Lagrange equations reproduce the Finsler geodesic equation. Using this general setup, we derived the modified geodesic equation for a first-order perturbation directly in the Lagrangian framework and then extended it to the n-polynomial case. From the derivation of the n-polynomial formula in both frameworks, we see that the two expressions differ only by a term that can be absorbed through a reparametrization of the curve. Based on this, we put forward a conjecture that for any perturbation function that is a homogeneous polynomial in the momenta, an antisymmetric structure of the form $(\nabla_m A^a - \nabla^a A_m)$ will always appear in the geodesic equation. This is consistent with all the examples derived in this work, each of which can be viewed as a special case of the homogeneous n-polynomial formulation. The antisymmetric term vanishes when the background vector field is a gradient of a scalar function, $A_a = \partial_a f$, and when no additional structures, such as the free-index A^a term, are present. In those cases, the modified trajectory reduces to the unperturbed geodesic up to a reparametrization. The same set of examples was analyzed in both frameworks, providing a consistent check of the results.

A detailed, step-by-step discussion of these findings is presented below.

We began by deriving the geodesic equation for the Hamiltonian $H(x,p) = g(p,p) + \epsilon\, g(p,p)A(p)$ in section IV. The right-hand side of equation (64) depends only on the covariant derivatives of the background vector field, and all velocity terms ($\dot{x}$) can be factored out, which means that we get a tensorial equation purely in the base manifold. This shows that if the background field A_a is the gradient of a scalar function, $A_a = \partial_a f$, the antisymmetric bracket $(\nabla_m A^a - \nabla^a A_m)$ vanishes. In that case, the modification disappears, leaving only a term proportional to $\dot{x}^a$, which can be removed by reparametrization. The resulting trajectories coincide with those of general relativity—illustrating that certain MDRs can reproduce standard geodesics up to a change of parameter. Moreover, the structure of the modified geodesic equation closely mirrors the equation of motion for a charged particle in an electromagnetic field on curved spacetime, which is particularly striking given that it arises here from a quantum-gravity–motivated background rather than a gauge field.

Next, I considered a general first order perturbation $H(x,p) = g(p,p) + \epsilon\, h(x,p)$ in section V. Following the same steps used in the previous section, the modified geodesic equation was obtained where the right-hand side naturally organizes into structures that would cancel all non-covariant contributions involving Christoffel symbols mimic the action of covariant derivatives. This formulation is powerful because it allows direct substitution of different $h(x,p)$ to see how modifications to the dispersion relation affect the geodesic equations.

As the first example I chose the case of a perturbation linear in the momenta, $h = A(p)$ (see V), the resulting geodesic correction (79) depended only on the covariant derivatives of the background vector field A_d and the modification appeared in an antisymmetric structure of the form $\nabla_m A^a - \nabla^a A_m$, which is exactly the same form of modification as we got for $h = g(p,p)A(p)$ (64) up to a parameterisation. This means that this geodesic equation turns out to have exactly the same form in both cases, despite the fact that the corresponding dispersion relations look very different in Minkowski spacetime (see equation (83)). Thus, two apparently unrelated modifications—one linear in momenta and one cubic—lead to identical modification to the geodesic equations. Then I set, $h = g(p,p)A(p)$ in (77) and found exactly same geodesic equation (A7) as (64), which serves as a consistency check for (77) . The situation changes when we consider a cubic perturbation of the form, $h = [A(p)]^3$

in section A 2. In this case, the geodesic equation no longer collapses to the same form as the earlier two examples. Instead, it contains an additional term carrying $A^{\text{free-index}}$. This surviving term cannot be eliminated through simple reparametrization, and it represents a qualitatively new constraint on the dynamics. Thus, although all three dispersion relations are polynomial functions of the momentum, they do not all give rise to the same type of geodesic modification. The first two are effectively equivalent modifications, while the cubic case introduces genuinely new behavior.

To check this further, we derive the geodesic equation for the perturbation function $h(x, \bar{p}) = \left(g(\bar{p}, \bar{p})\right)^r, [A(\bar{p})]^s$. This case also generalizes the string-theory–inspired model $h = [A(\bar{p})]^n$. From the resulting geodesic equation, we observe that three different structures appear: the antisymmetric structure $(\nabla_m A^a - \nabla^a A_m)$, the $A^{\text{free-index}}$ term, and a term involving A^a. Whether the A^a term survives depends on s, i.e. on the exponent of $A(p)$, finally, contributions multiplied by $\dot{x}$ term) that can be absorbed through a reparameterization of the curve and do not alter the physical trajectory itself. We also derive the $h = [A(\bar{p})]^n$ case separately in Appendix A 3. There we also see that the A^a term disappears when the power of $A(\bar{p})$, n is greater than one. This explains why the $g(p, p)A(p)$ case and the linear $A(p)$ case share the same geodesic structure, while $A(\bar{p})^3$ introduce genuinely new features that cannot be removed through reparametrization.

I also examined the case $h = \left(g(\bar{p}, \bar{p}) - [A(\bar{p})]^2\right)^N$ in section A 4, which corresponds to the modified dispersion relation describing D-particle backgrounds and vacuum refractive index models. For this case also, one can again clearly identify the same three distinct features.

We then move on to the geodesic equation for a modified dispersion relation containing an n-th order polynomial term homogeneous in the momenta in section V B. This derivation encompasses all the previous examples as special cases. It not only reproduces the simpler polynomial deformations but also unifies more complex string-inspired models, such as those describing D-particle backgrounds and vacuum refractive indices. Later, in the Lagrangian framework, we revisit this example and derive the geodesic equation using the n-polynomial formula, finding that the results in both pictures are almost identical up to a reparametrization. This confirms that the compact n-polynomial formula captures the essential structure of all homogeneous first-order perturbations. It also helps us understand why the antisymmetric term $(\nabla_m A^a - \nabla^a A_m)$ consistently appears in the geodesic equations we studied.

Having established the general results in the Hamiltonian framework, we next move to the Lagrangian formulation. The Helmholtz action is used to obtain the corresponding Finsler function $F(\dot{x}.\dot{x})$ on the tangent bundle from $H(x,p)$, which is related to the Lagrangian as $L = F^2$. The Euler–Lagrange equations derived from this Lagrangian then take the form of the Finsler geodesic equation, providing an equivalent description of the particle trajectories in velocity space. Using this setup, we derive the modified geodesic equation for a first-order perturbation directly within the Lagrangian picture. Having this general setup in hand, I went through explicit examples within the Lagrangian formalism. I solved specifically both the linear case $h = A(\bar{p})$ in section VII and the cubic case $h = g(p,p)A(p)$ in section B 2 and the more general $g(p,p)^r A(p)^s$ case in section B 3, calculating their geodesic equations via the Lagrangian geodesic formula. On initial inspection, the results seemed different from the Hamiltonian calculations. But on closer inspection, it was seen that the extra terms in the case of the Lagrangian are always parallel $\dot{x}^a$, which means they can be absorbed by a reparametrization. And the geodesic action is reparameterization-invariant, and so the geometry of the path is unchanged.

Using the geodesic equation (128), the n-polynomial formula for the geodesic equation in the Lagrangian framework is derived in section VII A and compared it to the corresponding expression obtained in the Hamiltonian picture (102). The two formulations differ by a single additional term proportional to $\dot{x}^q$, which is parallel to the velocity and can therefore be absorbed by a reparametrization of the curve. Apart from this term, the n-polynomial expressions in both frameworks matches structurally. Therefore, we reach the broad conclusion that for any homogeneous n-th order polynomial perturbation, the Hamiltonian and Lagrangian frameworks differ only by a gauge freedom associated with curve reparametrization. Using the Lagrangian n-polynomial formula, I then derived the geodesic equations for some examples to do a consistency check. In particular, both cubic cases—$h = g(p,p)A(p)$ in section B 4 b and $h = [A(p)]^3$ in section B 4 a, where I had already derived the geodesic equation for them using equation (128). I also derive the geodesic equations for the string-theory–inspired modifications $A(p)^n$ in section B 4 c and $(g(p,p) - (A(p))^2)^N$ in section VII B to show how the calculation might look for more complex cases. All reproduce the same structures as in the Hamiltonian framework once reparametrization freedom is taken into account.

At the end of this analysis, I would like to note that both the general geodesic equation for a first-order perturbation and the n-polynomial master formula in the Lagrangian picture were originally derived by my supervisor, Dr. Christian Pfeifer, whose insights made this extension and comparison possible.

To summerise, the most general result in this work is the geodesic equation derived for a *general first-order perturbation*, $H(x,p) = g(p,p) + \epsilon\, h(x,p)$. The n-polynomial formula is a *specialization* obtained when the perturbation function $h(x,p)$ is assumed to be a homogeneous polynomial in the momenta. Across all homogeneous examples studied, we observe that the antisymmetric structure $(\nabla_m A^a - \nabla^a A_m)$ always appears in the geodesic equation. While we do not prove this in full generality here, this motivates the conjecture that such an antisymmetric term in the geodesic equation is an inherent feature of *all* homogeneous polynomial perturbations.

Outlook and future directions: The results presented here open several possible directions for future work. A natural next step would be a full classification of the geodesic equations obtained from different classes of Hamiltonians, similar to the classification schemes known for α–β Finsler geometries. This would help clarify how various MDR-induced geometries relate to established Finsler structures. Another promising route is to apply the general framework developed here to compute explicit observable effects—such as redshift, lensing, or time-delay corrections—in curved backgrounds beyond the simple examples discussed. For the cases where $h = \cdots A(p)^n$ with $n > 1$, where the free-index term A^q survives, a natural next step is to test the solvability of the resulting differential equation by setting the right hand side of the the geodesic equation to zero; when solvable, one can solve it explicitly to extract conditions on the background field A_a under which the motion reduces to the general relativistic geodesics.

One important line of future work would be the systematic comparison between quantum-gravity-motivated MDRs-induced deviations from geodesic motion and those due to classical effects such as radiation-reaction or self-force. In the classical limit, force tensors such as gravitational self-force or electromagnetic radiation-reaction terms $(F^{\mu}_{\text{self}},\ F^{\mu}_{\text{RR}})$—enter the equations of motion and produce small but conceptually significant deviations from background geodesic. It is possible that a study into whether the non-geodesic motion being seen is due in large part to Planck-scale quantum corrections, classical backreaction, or a combination of the two could give insight into the mechanisms behind these phenomena. Comparing the two would clarify physical motivations for deviations in the geodesic equation and might offer observational techniques for distinguishing classical and quantum-gravity signatures in future experiments [14] [13].

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APPENDIX

This appendix provides additional derivations and supporting computations complementing the main text. While the overall discussion of the thesis focuses on conceptual development and exposition of major results, following sections possess the full technical details of certain intermediate steps along with certain explicit examples which helped to form the train of reasoning.

We begin by computing the coordinate derivative of $g(p,p)$ that we will repeatedly use in the main text. The subsequent sections provide the complete derivations of the geodesic equations for a variety of perturbation functions, initially in the Hamiltonian and then in the Lagrangian formalisms. The results presented here are not considered as the main results of the thesis but serve to clarify the sequence of calculations which determined the thesis's structure. All of the results derived here are referenced elsewhere in the main text, where only their final forms were provided.
Now compute the coordinate derivative:

$$\partial_a g(p,p) = \partial_a g^{cd} p_c p_d \tag{154}$$

We know that

$$\begin{aligned}
&\nabla_a g^{cd} = 0 \\
&\Rightarrow \frac{\partial g^{cd}}{\partial x^a} + g^{ci}\Gamma^d_{ai} + g^{di}\Gamma^c_{ai} = 0 \\
&\Rightarrow \frac{\partial g^{cd}}{\partial x^a} = -(g^{ci}\Gamma^d_{ai} + g^{di}\Gamma^c_{ai}) \\
&\Rightarrow \frac{\partial g^{cd}}{\partial x^a} p_c p_d = -(g^{ci}\Gamma^d_{ai} p_c p_d + g^{di}\Gamma^c_{ai} p_c p_d) = 2p^i \Gamma^d_{ai} p_d \\
&\Rightarrow \partial_a g(p,p) = -2\, p^c \Gamma^d_{ac} p_d
\end{aligned} \tag{155}$$

Appendix A: Hamiltonian Framework: Examples with Specific Perturbation Functions

We here explicitly solve the geodesic equations for some specific perturbation functions in the Hamiltonian framework. The derivation in all these cases follows from the general first-order perturbative geodesic equation (77) derived in the main text. The purpose here is to show the application of the general formula to specific models. We begin with the mixed cubic case $h(x,p) = g(p,p)A(p)$ and $A(p)^3$ and finally two string-theory–inspired models.

1. The geodesic equation for $h(x,p) = g(p,p)A(p)$

In this section, we derive the geodesic equation corresponding to $h(x,p) = g(p,p)A(p)$ by substituting this specific form of h into equation (77). Since the geodesic equation for this perturbation function was already obtained in section IV, the present derivation serves as a consistency check for equation (77). We begin with the choice of deformation function:

$$h(x,p) = (A^q p_q)\, g^{mn} p_m p_n \tag{A1}$$

To derive the geodesic equation we compute the following derivatives:

The p derivative:

$$\begin{aligned}\bar{\partial}^a h &= A^a\, g(p,p) + 2A(p)\, g^{an} p_n \\ &= A^a \cdot \frac{1}{4} g(\dot{x},\dot{x}) + \frac{1}{2} A_q \dot{x}^q\, g^{an} \dot{x}_n \\ &= \frac{1}{4} g(\dot{x},\dot{x}) A^a + \frac{1}{2} A(\dot{x})\, \dot{x}^a\end{aligned} \tag{A2}$$

The p derivative of equation(A2) is:

$$\begin{aligned}\bar{\partial}^b \bar{\partial}^a h &= 2A^a g^{bn} p_n + 2A^b g^{an} p_n + 2g^{ab}(A^q p_q) \\ &= A^a g^{bn} \dot{x}_n + A^b g^{an} \dot{x}_n + g^{ab} A_q \dot{x}^q \\ &= A^a \dot{x}^b + A^b \dot{x}^a + g^{ab} A(\dot{x})\end{aligned} \tag{A3}$$

The x derivative of equation(A2):

$$\begin{aligned}\partial_b \bar{\partial}^a h &= \partial_b \left(A^a g(p,p)\right) + \partial_b \left(A(p) g^{an} p_n\right) \\ &= \left(\nabla_b A^a - \Gamma^a_{bd} A^d\right) g(p,p) - 2A^a\, p^c \Gamma^d_{bc} p_d \\ &\quad + 2\left[p_d \left(\nabla_b A^d - \Gamma^d_{bc} A^c\right)\right] g^{an} p_n - 2A(p) \left(\Gamma^a_{bm} g^{mn} + \Gamma^n_{bm} g^{am}\right) p_n\end{aligned} \tag{A4}$$

Substituting $p^a = \frac{1}{2}\dot{x}^a$, $g(p,p) = \frac{1}{4} g(\dot{x},\dot{x})$, and $A^q p_q = \frac{1}{2} A_q \dot{x}^q$, we obtain:

$$\begin{aligned}\partial_b \bar{\partial}^a h &= \frac{1}{4} g(\dot{x},\dot{x}) \left(\nabla_b A^a - \Gamma^a_{bd} A^d\right) - \frac{1}{2} A^a\, \dot{x}^c\, \Gamma^d_{bc} \dot{x}_d \\ &\quad + \frac{1}{2} \dot{x}_d \left(\nabla_b A^d - \Gamma^d_{bc} A^c\right) \dot{x}^a - \frac{1}{2} A(\dot{x})\, \Gamma^a_{bm} \dot{x}^m - \frac{1}{2} A(\dot{x})\, \Gamma^n_{bm} g^{am} \dot{x}_n\end{aligned} \tag{A5}$$

And the x derivative of h is:

$$\partial_b h = \frac{1}{8}\, g(\dot{x},\dot{x})\, \dot{x}_q\, \nabla_b A^q - \frac{1}{8}\, g(\dot{x},\dot{x})\, A^d\, \dot{x}_q\, \Gamma^q_{bd} - \frac{1}{4}\, A_m\, \dot{x}^m\, \dot{x}^c\, \Gamma^d_{bc}\, \dot{x}_d \tag{A6}$$

We now substitute these components into the modified geodesic equation(77), and the right hand side of the geodesic equation simplifies to:

$$\begin{aligned}\ddot{x}^a + \Gamma^a{}_{bd}\dot{x}^d\dot{x}^b &= \epsilon\left[-\frac{1}{4}g(\dot{x},\dot{x})\,\dot{x}^m\,\nabla^a A_m + \frac{1}{4}g(\dot{x},\dot{x})\,\dot{x}^b\nabla_b A^a + \frac{1}{2}\dot{x}^a\dot{x}^d\dot{x}^b\nabla_b A_d\right]\\ &= \frac{\epsilon}{4}\dot{x}^b\,\Big(g(\dot{x},\dot{x})\big(\nabla_b A^a - \nabla^a A_b\big)\; + 2\dot{x}^a\dot{x}^d\dot{x}^b\nabla_b A_d\Big)\end{aligned} \tag{A7}$$

Therefore, we get the exact same equation (64).

2. The geodesic equation for $h = [A(p)]^3$

In this section, we investigate the geodesic equation corresponding to the perturbation function

$$h = (A^q p_q)(A^m p_m)(A^n p_n) = [A(p)]^3$$

This calculation is carried out to understand why the linear case $h = A(p)$ and the mixed cubic case $h = g(p,p)A(p)$ lead to identical geodesic structures, and to test whether this similarity extends to the purely cubic case as well. By doing so, we aim to identify whether the antisymmetric structure observed in the previous cases continues to appear here, or if new geometric features emerge.

To derive the geodesic equation we compute the following derivatives of h:

The p derivative:

$$\begin{aligned}\bar{\partial}^a h = \frac{\partial h}{\partial p_a} &= 3(A^q p_q)^2 A^a\\ &= 3\left(\frac{1}{2}A_q\dot{x}^q\right)^2 A^a = \frac{3}{4}A(\dot{x})^2 A^a\end{aligned} \tag{A8}$$

The x derivative:

$$\begin{aligned}
\partial_b h = \frac{\partial h}{\partial x^b} &= 3(A^q p_q)^2 \left(p_m \nabla_b A^m - A^m \Gamma^n_{bm} p_n\right) \\
&= 3\left(\frac{1}{2}A_q \dot{x}^q\right)^2 \left(\frac{1}{2}\dot{x}_m \nabla_b A^m - \frac{1}{2}A^m \Gamma^n_{bm}\dot{x}_n\right) \\
&= \frac{3}{4}A(\dot{x})^2 \left(\frac{1}{2}\dot{x}_m \nabla_b A^m - \frac{1}{2}A^m \Gamma^n_{bm}\dot{x}_n\right) \\
&= \frac{3}{8}A(\dot{x})^2 \left(\dot{x}_m \nabla_b A^m - A^m \Gamma^n_{bm}\dot{x}_n\right)
\end{aligned} \tag{A9}$$

The p derivative of $\bar{\partial}^a h$:

$$\begin{aligned}
\bar{\partial}^b \bar{\partial}^a h = \frac{\partial}{\partial p_b}\left(\frac{\partial h}{\partial p_a}\right) &= \frac{\partial}{\partial p_b}\left(3(A^q p_q)^2 A^a\right) \\
&= 6(A^q p_q) A^a A^b \\
&= 6\left(\frac{1}{2}A_q \dot{x}^q\right) A^a A^b = 3A(\dot{x}) A^a A^b
\end{aligned} \tag{A10}$$

the x derivative of $\bar{\partial}^a h$:

$$\begin{aligned}
\partial_b \bar{\partial}^a h &= \partial_b \left(3(A^q p_q)^2 A^a\right) \\
&= 6(A^q p_q)\left(p_m \nabla_b A^m - A^m \Gamma^n_{bm} p_n\right) A^a + 3(A^q p_q)^2 \left(\nabla_b A^a - \Gamma^a_{bd} A^d\right) \\
&= 6\left(\frac{1}{2}A_q \dot{x}^q\right)\left(\frac{1}{2}\dot{x}_m \nabla_b A^m - \frac{1}{2}A^m \Gamma^n_{bm}\dot{x}_n\right) A^a + 3\left(\frac{1}{2}A_q \dot{x}^q\right)^2 \left(\nabla_b A^a - \Gamma^a_{bd} A^d\right) \\
&= \frac{3}{2}A(\dot{x})\left(\dot{x}_m \nabla_b A^m - A^m \Gamma^n_{bm}\dot{x}_n\right) A^a + \frac{3}{4}A(\dot{x})^2 \left(\nabla_b A^a - \Gamma^a_{bd} A^d\right)
\end{aligned} \tag{A11}$$

Finally, substituting these expressions into equation(77), the first part becomes:

$$\begin{aligned}
-g^{ab}\left(2\partial_b h + \dot{x}_d \Gamma^d{}_{bc} \bar{\partial}^c h\right) &= -g^{ab}\left(2 \cdot \frac{3}{8}A(\dot{x})^2 \left(\dot{x}_m \nabla_b A^m - A^m \Gamma^n_{bm}\dot{x}_n\right) + \dot{x}_d \Gamma^d{}_{bc} \cdot \frac{3}{4}A(\dot{x})^2 A^c\right) \\
&= -g^{ab}\left(\frac{3}{4}A(\dot{x})^2 \left(\dot{x}_m \nabla_b A^m - A^m \Gamma^n_{bm}\dot{x}_n\right) + \frac{3}{4}A(\dot{x})^2 \dot{x}_d \Gamma^d{}_{bc} A^c\right) \\
&= -\frac{3}{4}A(\dot{x})^2 g^{ab}\left(\dot{x}_m \nabla_b A^m\right)
\end{aligned} \tag{A12}$$

The second part:

$$
\begin{aligned}
&\dot{x}^b\partial_b\bar{\partial}^a h + \frac{1}{2}\dot{x}^c\dot{x}^d\Gamma_{dbc}\bar{\partial}^b\bar{\partial}^a h + \dot{x}^b\Gamma^a{}_{bd}\bar{\partial}^d h \\
&= \dot{x}^b\left[\frac{3}{2}A(\dot{x})\left(\dot{x}_m\nabla_b A^m - A^m\Gamma^n_{bm}\dot{x}_n\right)A^a + \frac{3}{4}A(\dot{x})^2\left(\nabla_b A^a - \Gamma^a_{bd}A^d\right)\right] \\
&\quad + \frac{1}{2}\dot{x}^c\dot{x}^d\Gamma_{dbc}\cdot 3A(\dot{x})A^bA^a + \dot{x}^b\Gamma^a{}_{bd}\cdot\frac{3}{4}A(\dot{x})^2A^d \\
&= \frac{3}{2}A(\dot{x})\dot{x}^b\dot{x}_m\nabla_b A^m A^a - \frac{3}{2}A(\dot{x})\dot{x}^bA^m\Gamma^n_{bm}\dot{x}_nA^a + \frac{3}{4}A(\dot{x})^2\dot{x}^b\nabla_bA^a - \frac{3}{4}A(\dot{x})^2\dot{x}^b\Gamma^a_{bd}A^d \\
&\quad + \frac{3}{2}A(\dot{x})\dot{x}^c\dot{x}^d\Gamma_{dbc}A^bA^a + \frac{3}{4}A(\dot{x})^2\dot{x}^b\Gamma^a{}_{bd}A^d \\
&= \frac{3}{2}A(\dot{x})\dot{x}^b\dot{x}_m\nabla_bA^mA^a + \frac{3}{4}A(\dot{x})^2\dot{x}^b\nabla_bA^a
\end{aligned}
\tag{A13}
$$

Therefore, the geodesic equation,

$$
\begin{aligned}
\ddot{x}^a + \Gamma^a{}_{bd}\dot{x}^d\dot{x}^b &= -\frac{3}{4}A(\dot{x})^2g^{ab}\dot{x}_m\nabla_bA^m + \frac{3}{2}A(\dot{x})\,\dot{x}^b\dot{x}_m\,\nabla_bA^m\,A^a + \frac{3}{4}A(\dot{x})^2\,\dot{x}^b\nabla_bA^a \\
&= \frac{3}{4}(A\cdot\dot{x})\,\dot{x}^m\Big((A\cdot\dot{x})(\nabla_mA^a - \nabla^aA_m) + 2A^a\dot{x}^d\nabla_dA_m\Big)
\end{aligned}
\tag{A14}
$$

3. The geodesic equation for for Microscopic Recoil Model

In this section, I derive the geodesic equation from the modified dispersion relation for the string-inspired microscopic recoil model. This is a special case of the perturbation function $h(x,p) = \left(g^{bc}(x)p_bp_c\right)^k\left(A^d(x)p_d\right)^r$ which we had derived in the main text V A. In the bottom part of that section, the theoretical motivation of this model can be found. We begin by taking so derivatives of the perturbation function for the geodesic equation

$$
\begin{aligned}
h &= \left(A^qp_q\right)^N = \prod_{i=1}^{N}A^{a_i}p_{a_i} \\
\bar{\partial}^a h &= \frac{\partial h}{\partial p_a} = N\left(A^qp_q\right)^{N-1}A^a \\
&= N\left(\frac{1}{2}A_q\dot{x}^q\right)^{N-1}A^a = \frac{N}{2^{N-1}}A(\dot{x})^{N-1}A^a
\end{aligned}
\tag{A15}
$$

$$
\begin{aligned}
\partial_b h &= \frac{\partial h}{\partial x^b} = N\left(A^qp_q\right)^{N-1}\left(p_m\nabla_bA^m - A^m\Gamma^n_{bm}p_n\right) \\
&= N\left(\frac{1}{2}A_q\dot{x}^q\right)^{N-1}\left(\frac{1}{2}\dot{x}_m\nabla_bA^m - \frac{1}{2}A^m\Gamma^n_{bm}\dot{x}_n\right) \\
&= \frac{N}{2^N}A(\dot{x})^{N-1}\left(\dot{x}_m\nabla_bA^m - A^m\Gamma^n_{bm}\dot{x}_n\right)
\end{aligned}
\tag{A16}
$$

$$\bar{\partial}^b \bar{\partial}^a h = \frac{\partial}{\partial p_b}\left(\bar{\partial}^a h\right) = \frac{\partial}{\partial p_b}\left(N\left(A^q p_q\right)^{N-1} A^a\right)$$
$$= N(N-1)\left(A^q p_q\right)^{N-2} A^a A^b = \frac{N(N-1)}{2^{N-2}} A(\dot{x})^{N-2} A^a A^b \tag{A17}$$

$$\begin{aligned}
\partial_b \bar{\partial}^a h &= \partial_b\left(\bar{\partial}^a h\right) = \partial_b\left(\frac{N}{2^{N-1}} A(\dot{x})^{N-1} A^a\right) \\
&= \frac{N}{2^{N-1}}\left[\partial_b\left(A(\dot{x})^{N-1}\right) A^a + A(\dot{x})^{N-1} \partial_b A^a\right] \\
&= \frac{N}{2^{N-1}}\left[(N-1) A(\dot{x})^{N-2}\left(\dot{x}_m \nabla_b A^m - A^m \Gamma^n_{bm} \dot{x}_n\right) A^a + A(\dot{x})^{N-1}\left(\nabla_b A^a - \Gamma^a_{bc} A^c\right)\right] \\
&= \frac{N}{2^{N-1}} A(\dot{x})^{N-2}\left[(N-1) A^a\left(\dot{x}_m \nabla_b A^m - A^m \Gamma^n_{bm} \dot{x}_n\right) + A(\dot{x})\left(\nabla_b A^a - \Gamma^a_{bc} A^c\right)\right]
\end{aligned} \tag{A18}$$

Now evaluating the bracketed expressions for the geodesic equation

$$\begin{aligned}
g^{ab}\left(2\,\partial_b h + \dot{x}_d\, \Gamma^d{}_{bc}\, \bar{\partial}^c h\right) &= g^{ab}\left[2 \cdot \frac{N}{2^N} A(\dot{x})^{N-1}\left(\dot{x}_m \nabla_b A^m - A^m \Gamma^n_{bm} \dot{x}_n\right) + \dot{x}_d \Gamma^d{}_{bc} \cdot \frac{N}{2^{N-1}} A(\dot{x})^{N-1} A^c\right] \\
&= \frac{N}{2^{N-1}} A(\dot{x})^{N-1}\left[g^{ab} \dot{x}_m \nabla_b A^m - g^{ab} A^m \Gamma^n_{bm} \dot{x}_n + g^{ab} \dot{x}_d \Gamma^d{}_{bc} A^c\right] \\
&= \frac{N}{2^{N-1}} A(\dot{x})^{N-1}\left[g^{ab} \dot{x}_m \nabla_b A^m - A^m \dot{x}^n \Gamma_{nm}{}^a + \dot{x}^d A^c \Gamma_{dc}{}^a\right] \\
&= \frac{N}{2^{N-1}} A(\dot{x})^{N-1} g^{ab} \dot{x}_m \nabla_b A^m
\end{aligned} \tag{A19}$$

$$\begin{aligned}
&\dot{x}^b \partial_b \bar{\partial}^a h + \frac{1}{2} \dot{x}^c \dot{x}^d \Gamma_{dbc} \bar{\partial}^b \bar{\partial}^a h + \dot{x}^b \Gamma^a{}_{bd} \bar{\partial}^d h \\
&= \frac{N}{2^{N-1}} A(\dot{x})^{N-2}\Bigg[(N-1) A^a \dot{x}^b\left(\dot{x}^m \nabla_b A_m - A^m \Gamma_{nbm} \dot{x}^n\right) + A(\dot{x}) \dot{x}^b\left(\nabla_b A^a - \Gamma^a_{bc} A^c\right) \\
&\qquad + (N-1) A^a A^b \dot{x}^c \dot{x}^d \Gamma_{dbc} + A(\dot{x}) \dot{x}^b \Gamma^a{}_{bd} A^d\Bigg] \\
&= \frac{N}{2^{N-1}} A(\dot{x})^{N-2}\Bigg[(N-1) A^a \dot{x}^b \dot{x}^m \nabla_b A_m + A(\dot{x}) \dot{x}^b \nabla_b A^a\Bigg]
\end{aligned} \tag{A20}$$

Then from the geodesic equation 77

$$
\begin{aligned}
\ddot{x}^a + \Gamma^a{}_{bd}\dot{x}^d\dot{x}^b &= \epsilon\Bigg[- g^{ab}\left(2\partial_b h + \dot{x}_d\Gamma^d{}_{bc}\bar{\partial}^c h\right) + \left(\dot{x}^b\partial_b\bar{\partial}^a h + \frac{1}{2}\dot{x}^c\dot{x}^d\Gamma_{dbc}\bar{\partial}^b\bar{\partial}^a h + \dot{x}^b\Gamma^a{}_{bd}\bar{\partial}^d h\right)\Bigg] \\
&= \epsilon\Bigg[- \frac{N}{2^{N-1}}A(\dot{x})^{N-1}g^{ab}\dot{x}_m\nabla_b A^m + \\
&\quad \frac{N}{2^{N-1}}A(\dot{x})^{N-2}\Bigg((N-1)A^a\dot{x}^b\dot{x}^m\nabla_b A_m + A(\dot{x})\dot{x}^b\nabla_b A^a\Bigg)\Bigg] \\
&= \epsilon\frac{N}{2^{N-1}}A(\dot{x})^{N-1}\Bigg[- \dot{x}^m\nabla^a A_m + A(\dot{x})^{-1}(N-1)A^a\dot{x}^b\dot{x}^m\nabla_b A_m + \dot{x}^m\nabla_m A^a\Bigg] \\
&= \epsilon\frac{N}{2^{N-1}}A(\dot{x})^{N-1}\dot{x}^m\Bigg[g^{ab}\left(\nabla_m A_b - \nabla_b A_m\right) + \frac{(N-1)A^a\dot{x}^b}{A(\dot{x})}\nabla_b A_m\Bigg] \qquad \text{(A21)}
\end{aligned}
$$

To check the consistency of this general formula, We can substitute N= 3 in equation(A21), and we see that we get the exact same expression as A14.

4. MDR for D-particle Backgrounds and Vacuum Refractive Indices

To explore the dynamical consequences of this string-inspired MDR, we now derive the corresponding geodesic equation by inserting $h = \left(g^{ab}p_a p_b - (A^q p_q)^2\right)^N$ into the general first-order perturbative framework (77). The theoretical motivation of this model can be found in VII B We start by taking the following derivatives of the perturbation function for the geodesic equation:

$$
\begin{aligned}
h &= \left(g^{ab}p_a p_b - (A^q p_q)^2\right)^N \\
\bar{\partial}^a h = \frac{\partial h}{\partial p_a} &= N\left(g^{mn}p_m p_n - (A^q p_q)^2\right)^{N-1}\left(2g^{ab}p_b - 2A^a A^q p_q\right) \\
&= N\left(\tfrac{1}{4}g(\dot{x},\dot{x}) - \tfrac{1}{4}A(\dot{x})^2\right)^{N-1}\left(2\,g^{ab}\cdot\tfrac{1}{2}\dot{x}_b - 2A^a\cdot\tfrac{1}{2}A(\dot{x})\right) \\
&= N\left(\tfrac{1}{4}(g(\dot{x},\dot{x}) - A(\dot{x})^2)\right)^{N-1}\left(\dot{x}^a - A^a A(\dot{x})\right). \qquad \text{(A22)}
\end{aligned}
$$

$$
\begin{aligned}
\bar\partial^b\bar\partial^a h &= \frac{\partial^2 h}{\partial p_b \partial p_a}\\
&= N(N-1)\left(g^{mn}p_m p_n - (A^q p_q)^2\right)^{N-2}(2g^{ac}p_c - 2A^a A^q p_q)\left(2g^{bd}p_d - 2A^b A^r p_r\right)\\
&\quad + N\left(g^{mn}p_m p_n - (A^q p_q)^2\right)^{N-1}\left(2g^{ab} - 2A^a A^b\right)\\
&= N(N-1)\left(\tfrac{1}{4}g(\dot x,\dot x) - \tfrac{1}{4}A(\dot x)^2\right)^{N-2}(\dot x^a - A^a A(\dot x))\left(\dot x^b - A^b A(\dot x)\right)\\
&\quad + N\left(\tfrac{1}{4}(g(\dot x,\dot x) - A(\dot x)^2)\right)^{N-1}\left(2g^{ab} - 2A^a A^b\right)
\end{aligned} \tag{A23}
$$

$$
\begin{aligned}
\partial_b h &= N\left(g^{mn}p_m p_n - (A^q p_q)^2\right)^{N-1}(\partial_b g^{mn}p_m p_n - 2A^q p_q \cdot \partial_b A^r p_r)\\
&= N\left(g^{mn}p_m p_n - (A^q p_q)^2\right)^{N-1}\left(-2p^c\Gamma^d_{bc}p_d - 2A(p)\left[p_r\nabla_b A^r - p_r\Gamma^r_{bq}A^q\right]\right)\\
&= N\left(\tfrac{1}{4}g(\dot x,\dot x) - \tfrac{1}{4}A(\dot x)^2\right)^{N-1}\left(-2\cdot\tfrac{1}{2}\dot x^c\,\Gamma^d_{bc}\cdot\tfrac{1}{2}\dot x_d - 2\cdot\tfrac{1}{2}A(\dot x)\cdot\left[\tfrac{1}{2}\dot x_r\nabla_b A^r - \tfrac{1}{2}\dot x_r\Gamma^r_{bq}A^q\right]\right)\\
&= N\left(\tfrac{1}{4}(g(\dot x,\dot x) - A(\dot x)^2)\right)^{N-1}\left(-\tfrac{1}{2}\dot x^c\Gamma^d_{bc}\dot x_d - \tfrac{1}{2}A(\dot x)\left[\dot x_r\nabla_b A^r - \dot x_r\Gamma^r_{bq}A^q\right]\right)
\end{aligned} \tag{A24}
$$

$$
\begin{aligned}
\partial_b\bar\partial^a h &= \partial_b\Big\{N\Big(g(p,p) - (A^q p_q)^2\Big)^{N-1}(2g^{ac}p_c - 2A^a A(p))\Big\}\\
&= N(N-1)\Big(g(p,p) - A(p)^2\Big)^{N-2}\partial_b\Big(g(p,p) - A(p)^2\Big)(2g^{ac}p_c - 2A^a A(p))\\
&\quad + N\Big(g(p,p) - A(p)^2\Big)^{N-1}(2(\partial_b g^{ac})p_c - 2(\partial_b A^a)A(p) - 2A^a\partial_b A(p))\\
&= N(N-1)\Big(\tfrac{1}{4}g(\dot x,\dot x) - \tfrac{1}{4}A(\dot x)^2\Big)^{N-2}\left(-\tfrac{1}{2}\,\dot x^c\Gamma^d{}_{bc}\dot x_d - \tfrac{1}{2}\,A(\dot x)\Big[\dot x^m\nabla_b A_m - \dot x^m\Gamma^n{}_{bm}A_n\Big]\right)\\
&\qquad\qquad\qquad\cdot(g^{ac}\dot x_c - A^a A(\dot x))\\
&\quad + N\Big(\tfrac{1}{4}\Big(g(\dot x,\dot x) - A(\dot x)^2\Big)\Big)^{N-1}\Big(-\Big(g^{ic}\Gamma^a{}_{bi} + g^{aj}\Gamma^c{}_{bj}\Big)\dot x_c - (\nabla_b A^a - \Gamma^a{}_{bc}A^c)A(\dot x)\\
&\qquad\qquad\qquad - A^a\Big[\dot x^m\nabla_b A_m - \dot x^m\Gamma^n{}_{bm}A_n\Big]\Big).
\end{aligned} \tag{A25}
$$

Now, evaluating the bracketed expressions for the geodesic equation (77)

$$
\begin{aligned}
&- g^{ab}\left(2\,\partial_b h + \dot{x}_d\,\Gamma^d{}_{bc}\,\bar{\partial}^c h\right)\\
&- g^{ab}\Big[2N\left(\tfrac{1}{4}(g(\dot{x},\dot{x}) - A(\dot{x})^2)\right)^{N-1}\left(-\tfrac{1}{2}\dot{x}^c\Gamma^d_{bc}\dot{x}_d - \tfrac{1}{2}A(\dot{x})\dot{x}_r\nabla_b A^r + \tfrac{1}{2}A(\dot{x})\dot{x}_r\Gamma^r_{bq}A^q\right)\\
&\quad +\dot{x}_d\Gamma^d_{bc}\cdot N\left(\tfrac{1}{4}(g(\dot{x},\dot{x}) - A(\dot{x})^2)\right)^{N-1}\left(\dot{x}^c - A^c A(\dot{x})\right)\Big]\\
&= -g^{ab}\cdot N\left(\tfrac{1}{4}(g(\dot{x},\dot{x}) - A(\dot{x})^2)\right)^{N-1}\cdot\Big[-\dot{x}^c\Gamma^d_{bc}\dot{x}_d - A(\dot{x})\dot{x}_r\nabla_b A^r + A(\dot{x})\dot{x}_r\Gamma^r_{bq}A^q + \dot{x}_d\Gamma^d_{bc}\dot{x}^c\\
&\quad - \dot{x}_d\Gamma^d_{bc}A^c A(\dot{x})\\
&= g^{ab}\cdot N\left(\tfrac{1}{4}(g(\dot{x},\dot{x}) - A(\dot{x})^2)\right)^{N-1} A(\dot{x})\dot{x}_r\nabla_b A^r\\
&= N\left(\tfrac{1}{4}(g(\dot{x},\dot{x}) - A(\dot{x})^2)\right)^{N-1} A(\dot{x})\dot{x}^r\nabla^a A_r
\end{aligned}
\tag{A26}
$$

The second one:

$$
\begin{aligned}
&\dot{x}^b\partial_b\bar{\partial}^a h + \frac{1}{2}\dot{x}^c\dot{x}^d\Gamma_{dbc}\bar{\partial}^b\bar{\partial}^a h + \dot{x}^b\Gamma^a{}_{bd}\bar{\partial}^d h\\
&= \tfrac{1}{2}N(N-1)\left(\tfrac{1}{4}g(\dot{x},\dot{x}) - \tfrac{1}{4}A(\dot{x})^2\right)^{N-2}\Big[-\dot{x}^b\dot{x}^a\dot{x}^c\Gamma^d_{bc}\dot{x}_d + A^a A(\dot{x})\,\dot{x}^b\dot{x}^c\Gamma^d_{bc}\dot{x}_d\\
&\qquad - A(\dot{x})\,\dot{x}^b\dot{x}^a\dot{x}^m\left(\nabla_b A_m - \Gamma^n_{bm}A_n\right) + A^a A(\dot{x})^2\,\dot{x}^b\dot{x}^m\left(\nabla_b A_m - \Gamma^n_{bm}A_n\right)\\
&\qquad + \dot{x}^b\dot{x}^a\dot{x}^c\dot{x}^d\Gamma_{dbc} - A^a A(\dot{x})\,\dot{x}^b\dot{x}^c\dot{x}^d\Gamma_{dbc} - A^b A(\dot{x})\,\dot{x}^a\dot{x}^c\dot{x}^d\Gamma_{dbc} + A^a A^b A(\dot{x})^2\,\dot{x}^c\dot{x}^d\Gamma_{dbc}\Big]\\
&\quad + N\left(\tfrac{1}{4}(g(\dot{x},\dot{x}) - A(\dot{x})^2)\right)^{N-1}\Big(\dot{x}^c\dot{x}^d\Gamma_{dc}{}^a - A^a A^b\,\dot{x}^c\dot{x}^d\Gamma_{dbc} + \dot{x}^d\dot{x}^b\Gamma^a{}_{bd} - A^d A(\dot{x})\,\dot{x}^b\Gamma^a{}_{bd}\\
&\qquad - \left(\dot{x}^b\dot{x}^i\Gamma^a_{bi} + \dot{x}^b\dot{x}_c\Gamma_{cb}{}^a\right) - \left(\nabla_b A^a - \Gamma^a_{bc}A^c\right)\dot{x}^b A(\dot{x}) - A^a\dot{x}^b\left(\dot{x}^m\nabla_b A_m - \dot{x}^m\Gamma^n_{bm}A_n\right)\Big)\\
&= \tfrac{1}{2}N(N-1)\left(\tfrac{1}{4}g(\dot{x},\dot{x}) - \tfrac{1}{4}A(\dot{x})^2\right)^{N-2}\left(-A(\dot{x})\,\dot{x}^b\dot{x}^a\dot{x}^m\nabla_b A_m + A^a A(\dot{x})^2\,\dot{x}^b\dot{x}^m\nabla_b A_m\right)\\
&\quad - N\left(\tfrac{1}{4}(g(\dot{x},\dot{x}) - A(\dot{x})^2)\right)^{N-1}\left(\dot{x}^b A(\dot{x})\,\nabla_b A^a + A^a\dot{x}^b\dot{x}^m\nabla_b A_m\right).
\end{aligned}
\tag{A27}
$$

Therefore, the geodesic equation(77) becomes,

$$
\begin{aligned}
\ddot{x}^a + \Gamma^a{}_{bd}\dot{x}^d\dot{x}^b = \epsilon\Bigg[& N\left(\tfrac{1}{4}(g(\dot{x},\dot{x}) - A(\dot{x})^2)\right)^{N-1} A(\dot{x})\dot{x}^r\nabla^a A_r \\
& + \tfrac{1}{2}N(N-1)\left(\tfrac{1}{4}g(\dot{x},\dot{x}) - \tfrac{1}{4}A(\dot{x})^2\right)^{N-2}\left(- A(\dot{x})\dot{x}^b\dot{x}^a\dot{x}^m\nabla_b A_m + A^a A(\dot{x})^2\dot{x}^b\dot{x}^m\nabla_b A_m\right) \\
& - N\left(\tfrac{1}{4}(g(\dot{x},\dot{x}) - A(\dot{x})^2)\right)^{N-1}\left(\dot{x}^b A(\dot{x})\nabla_b A^a + A^a\dot{x}^b\dot{x}^m\nabla_b A_m\right)\Bigg] \\
= \epsilon\Bigg[& N\left(\tfrac{1}{4}(g(\dot{x},\dot{x}) - A(\dot{x})^2)\right)^{N-1}\left(A(\dot{x})\dot{x}^b(\nabla^a A_b - \nabla_b A^a) - A^a\dot{x}^b\dot{x}^m\nabla_b A_m\right) \\
& + \tfrac{1}{2}N(N-1)\left(\tfrac{1}{4}g(\dot{x},\dot{x}) - \tfrac{1}{4}A(\dot{x})^2\right)^{N-2}\left(A(\dot{x})\,\dot{x}^b\,\dot{x}^m\,(-\dot{x}^a + A^a A(\dot{x}))\,\nabla_b A_m\right)\Bigg]
\end{aligned}
\tag{A28}
$$

Appendix B: Lagrangian Framework: Examples with Specific Perturbation Functions

In this section, I evaluate the geodesic equation for most of the examples that we did in the Hamiltonian framework, and I do so using the geodesic equation developed in the Lagrangian framework for a general first-order perturbation (128). The procedure involves computing the required derivatives of the perturbation function and substituting them into the general expression (128). Later, we derive the n-polynomial formula from this equation and would recompute some of the examples that we had derived using (128) which would serve as a consistency check for the n-polynomial formula. Then, I would use (147) to obtain the geodesic equations for the remaining perturbation functions discussed in the subsequent examples.

1. The geodesic equation for $h = [A(p)]^3$

We begin with the perturbation function:

$$
h(x,p) = \left(A^q(x)p_q\right)^3 \;\;\Rightarrow\;\; h(x,\dot{x}) = \left(A^q(x)\,\bar{p}_q(x,\dot{x})\right)^3, \tag{B1}
$$

with

$$
\bar{p}_a = m\frac{g_{ab}(x)\dot{x}^b}{\ell}, \qquad \ell = \sqrt{g_{ij}(x)\dot{x}^i\dot{x}^j}. \tag{B2}
$$

The geodesic equation:

$$\ddot{x}^q + \Gamma^q{}_{ij}\dot{x}^i\dot{x}^j = \frac{\epsilon}{2m^2}\left(g(\dot{x},\dot{x})g^{qb}(\dot{x}^a\delta_a\dot{\partial}_b h - \Gamma^c{}_{bj}\dot{x}^j\dot{\partial}_c h - \delta_b h) + 2\dot{x}^q\dot{x}^a\delta_a h\right) \tag{B3}$$

First,

$$\begin{aligned}\delta_a h &= 3(A^q\bar{p}_q)^2\,\delta_a(A^c\bar{p}_c)\\ &= 3(A^q\bar{p}_q)^2\left(\bar{p}_c\nabla_a A^c\right)\end{aligned} \tag{B4}$$

Next,

$$\begin{aligned}\dot{\partial}_b h &= 3(A^q\bar{p}_q)^2\,A^c\dot{\partial}_b\bar{p}_c\\ &= 3(A^q\bar{p}_q)^2\,A^c\left(\frac{m}{\ell}\left(g_{cb} - \frac{\dot{x}_c\dot{x}_b}{\ell^2}\right)\right).\end{aligned} \tag{B5}$$

For the mixed derivative we compute

$$\begin{aligned}\delta_a\dot{\partial}_b h &= 3(A^q\bar{p}_q)^2\Big[\frac{m}{\ell}\partial_a A^c\left(g_{bc} - \frac{\bar{p}_b\bar{p}_c}{m^2}\right) + \frac{m}{\ell}A^c\left(\Gamma_{bac} + \Gamma_{cab} - \frac{\bar{p}_d\Gamma^d{}_{ab}\bar{p}_c + \bar{p}_b\bar{p}_d\Gamma^d{}_{ac}}{m^2}\right)\Big]\\ &\quad + 6(A^q\bar{p}_q)\,(\delta_a(A^r\bar{p}_r))\,A^c\left(\frac{m}{\ell}\left(g_{cb} - \frac{\dot{x}_c\dot{x}_b}{\ell^2}\right)\right)\\ &= 3(A^q\bar{p}_q)^2\Big[\frac{m}{\ell}\partial_a A^c\left(g_{bc} - \frac{\dot{x}_b\dot{x}_c}{\ell^2}\right) + \frac{m}{\ell}A^c\left(\Gamma_{bac} + \Gamma_{cab} - \frac{\dot{x}_d\Gamma^d{}_{ab}\dot{x}_c + \dot{x}_b\dot{x}_d\Gamma^d{}_{ac}}{\ell^2}\right)\Big]\\ &\quad + 6\frac{m^2}{\ell^2}\,(A_q\dot{x}^q)\,(\dot{x}_r\nabla_a A^r)\,A^c\left(\frac{m}{\ell}\left(g_{cb} - \frac{\dot{x}_c\dot{x}_b}{\ell^2}\right)\right)\end{aligned} \tag{B6}$$

These terms can be combined into

$$\begin{aligned}&\dot{x}^a\delta_a\dot{\partial}_b h - \Gamma^c{}_{bj}\dot{x}^j\,\dot{\partial}_c h\\ &= 3\,\frac{m^2}{\ell^2}\,(A(\dot{x}))^2\,\dot{x}^a\left[\frac{m}{\ell}\,\partial_a A^c\left(g_{bc} - \frac{\dot{x}_b\dot{x}_c}{\ell^2}\right) + \frac{m}{\ell}\,A^c\left(\Gamma_{bac} + \Gamma_{cab} - \frac{\dot{x}_d\Gamma^d{}_{ab}\dot{x}_c + \dot{x}_b\dot{x}_d\Gamma^d{}_{ac}}{\ell^2}\right)\right]\\ &\quad + 6\,\frac{m^2}{\ell^2}\,(A(\dot{x}))\,(\dot{x}_r\nabla_a A^r)\,A^c\,\dot{x}^a\left(\frac{m}{\ell}\left(g_{bc} - \frac{\dot{x}_b\dot{x}_c}{\ell^2}\right)\right)\\ &\quad - \Gamma^c{}_{bj}\dot{x}^j\left[3\,\frac{m^2}{\ell^2}\,(A(\dot{x}))^2\,A^m\left(\frac{m}{\ell}\left(g_{cm} - \frac{\dot{x}_m\dot{x}_c}{\ell^2}\right)\right)\right]\\ &= 3\,\frac{m^3}{\ell^3}\,(A(\dot{x}))^2\left(g_{bc} - \frac{\dot{x}_b\dot{x}_c}{\ell^2}\right)\dot{x}^a\nabla_a A^c + 6\,\frac{m^3}{\ell^3}\,(A(\dot{x}))\,(\dot{x}_r\nabla_a A^r)\,\dot{x}^a\left(g_{bc} - \frac{\dot{x}_b\dot{x}_c}{\ell^2}\right)A^c\end{aligned} \tag{B7}$$

Hence the whole geodesic equation (128) becomes

$$\ddot{x}^q + \Gamma^q{}_{ij}\dot{x}^i\dot{x}^j = \frac{\epsilon}{2m^2}\Big(g(\dot{x},\dot{x})g^{qb}[3\,\frac{m^3}{\ell^3}(A(\dot{x}))^2\left(g_{bc} - \frac{\dot{x}_b\dot{x}_c}{\ell^2}\right)\dot{x}^a\nabla_a A^c \tag{B8}$$

$$+ 6\,\frac{m^3}{\ell^3}(A(\dot{x}))\,(\dot{x}_r\nabla_a A^r)\dot{x}^a\left(g_{bc} - \frac{\dot{x}_b\dot{x}_c}{\ell^2}\right)A^c \tag{B9}$$

$$- 3\frac{m^3}{\ell^3}\,(A(\dot{x}))^2\left(\dot{x}_c\nabla_b A^c\right)] + 6\dot{x}^q\dot{x}^a\cdot\frac{m^3}{\ell^3}\,(A(\dot{x}))^2\left(\dot{x}_c\nabla_a A^c\right)\Big)$$

$$= \frac{\epsilon m}{2\ell}\Big\{3(A(\dot{x}))^2\,\dot{x}^a\big(\nabla_a A^q - \nabla^q A_a\big) + 6(A(\dot{x}))\,(\dot{x}_r\nabla_a A^r)\dot{x}^a\Big(A^q - \frac{(A(\dot{x}))}{\ell^2}\dot{x}^q\Big)$$

$$+ 3\,\frac{(A(\dot{x}))^2}{\ell^2}\,\dot{x}^q\dot{x}^a\dot{x}_c\nabla_a A^c\Big\}$$

$$= \frac{\epsilon m}{2\ell}\Big\{3(A(\dot{x}))^2\,\dot{x}^a\big(\nabla_a A^q - \nabla^q A_a\big) + 6(A(\dot{x}))\,\dot{x}^a\dot{x}^r\nabla_a A_r\,A^q - 3\,\frac{(A(\dot{x}))^2}{\ell^2}\,\dot{x}^q\,\dot{x}^a\dot{x}^k\,\nabla_a A_k\Big\} \tag{B10}$$

In the Hamiltonian picture, the geodesic equation we had obtained was

$$\ddot{x}^a + \Gamma^a{}_{bd}\dot{x}^d\dot{x}^b = \frac{3}{4}(A\cdot\dot{x})\,\dot{x}^m\Big((A\cdot\dot{x})(\nabla_m A^a - \nabla^a A_m) + 2A^a\dot{x}^d\nabla_d A_m\Big) \tag{B11}$$

Although at first glance it looks a bit different than (B10), we can change the parametrisation $\tau \to s(\tau)$ so that

$$\dot{x}(s(\tau)) = \frac{d}{d\tau}x(s(\tau)) = \frac{dx}{ds}\frac{ds}{d\tau} = x'\dot{s}\,,$$
$$\ddot{x}(s(\tau)) = x'\ddot{s} + x''\dot{s}^2\,, \tag{B12}$$

which turns the geodesic equation to

$$x''^q + \Gamma^q{}_{ij}x'^i x'^j = \frac{3}{2}\cdot\frac{\epsilon m}{\ell}\Big((A\cdot x')^2\,x'^k(\nabla_k A^q - \nabla^q A_k)\big)\,+ 2\,A^q\,(A\cdot x')\,x'^c\,x'^b\nabla_b A_c\Big)$$
$$- x'^q\Big(\frac{3}{2}\,\frac{\epsilon m}{\ell^3}x'^a(A\cdot x')^2\,(\nabla_a A_k)\,x'^k + \frac{\ddot{s}}{\dot{s}^2}\Big) \tag{B13}$$

The last term vanishes if one choose a parametrization of the curve such that

$$\frac{\ddot{s}}{\dot{s}^2} = -\frac{3}{2}\,\frac{\epsilon m}{\ell^3}x'^a(A\cdot x')^2\,(\nabla_a A_k)\,x'^k\,, \tag{B14}$$

which that turns the geodesic equation closer to the one we get using Hamiltonian framework.

We see that geodesic equation of GR is re-obtained in both cases if

$$\Big((A\cdot\dot{x})(\nabla_m A^a - \nabla^a A_m) + 2A^a\dot{x}^d\nabla_d A_m\Big) = 0\,. \tag{B15}$$

2. The geodesic equation for $h(x,p) = g(p,p)A(p)$

We start by taking some derivatives of the perturbation function and then inserting them in equation (128)

$$
\begin{aligned}
h &= g(\bar{p},\bar{p})\, A(\bar{p}) \\
h(x,\dot{x}) &= \frac{m^3}{\ell}\, A_b(x)\, \dot{x}^b \\
\dot{\partial}_b h \; &= \; \frac{\partial}{\partial \dot{x}^b}\left(\frac{m^3}{\ell}\, A_q\, \dot{x}^q\right) \; = \; \frac{m^3}{\ell}\left(A_b - \frac{(A_q \dot{x}^q)\, \dot{x}_b}{\ell^2}\right) \\
\delta_a h &= m^2\, p_q\, \nabla_a A^q = m^3\, \frac{\dot{x}_q}{\ell}\, \nabla_a A^q. \\
\delta_a \dot{\partial}_b h &= \frac{m^3}{\ell}\left[\partial_a A_b - \frac{\dot{x}_b\, \dot{x}^k}{\ell^2}\, \nabla_a A_k - \frac{A{\cdot}\dot{x}}{\ell^2}\, \Gamma_{cab}\, \dot{x}^c\right].
\end{aligned}
\tag{B16}
$$

Evaluating the following expression:

$$
\dot{x}^a \delta_a \dot{\partial}_b h - \Gamma^c{}_{bj} \dot{x}^j \dot{\partial}_c h = \frac{m^3}{\ell}\left[\dot{x}^a \nabla_a A_b - \frac{\dot{x}_b}{\ell^2}\, \dot{x}^a \dot{x}^k \nabla_a A_k\right]
\tag{B17}
$$

Substituting everything in (128):

$$
\begin{aligned}
\ddot{x}^q + \Gamma^q{}_{ij} \dot{x}^i \dot{x}^j &= \frac{\epsilon}{2m^2}\, \frac{m^3}{\ell}\Big[\ell^2\, \dot{x}^a \nabla_a A^q - \ell^2\, \dot{x}^a \nabla^q A_a - \dot{x}^q\, \dot{x}^a \dot{x}^k \nabla_b A_k + 2\dot{x}^q\, \dot{x}^a \dot{x}^b \nabla_a A_b\Big] \\
&= \frac{\epsilon}{2m^2}\, \frac{m^3}{\ell}\Big[\ell^2\Big(\dot{x}^a \nabla_a A^q - \dot{x}^a \nabla^q A_a\Big) + \dot{x}^q\, \dot{x}^a \dot{x}^k \nabla_a A_k\Big] \\
&= \frac{\epsilon}{2m^2}\, \frac{m^3}{\ell} \dot{x}^a\Big[\ell^2\Big(\nabla_a A^q - \nabla^q A_a\Big) + \dot{x}^q\, \dot{x}^k \nabla_a A_k\Big] \\
&= \frac{\epsilon\, m}{2\,\ell}\, \dot{x}^a\Big[\Big(\nabla_a A^q - \nabla^q A_a\Big) + \dot{x}^q\, \dot{x}^k \nabla_a A_k\Big]
\end{aligned}
\tag{B18}
$$

which has a very similar expression to the one we had obtained in the Hamiltonian picture (A7) except for the last term which can be absorbed by reparameterization anyway.

We can re-parameterize from $\tau \to s(\tau)$ so that we re-obtain the Hamiltonian picture geodesic equation, where $s(\tau)$ has to satify

$$
\ddot{s}/\dot{s}^2 = \frac{\epsilon m}{2\ell}\, \dot{x}^a \dot{x}^k \nabla_a A_k
$$

This means GR geodesic is re-obtained whenever $\dot{x}^a g^{qb}(\nabla_a A_b - \nabla_b A_a) = 0$.

3. The geodesic equation for $g(p,p)^r A(p)^s$

In this section I derive the geodesic equation for the perturbation function $h(x,p) = \left(g^{bc}(x)p_b p_c\right)^r \left(A^d(x)p_d\right)^s$, where

$$\bar{p}_b = \frac{m}{\ell}, g_{bc}\dot{x}^c, \qquad \ell^2 = g(\dot{x},\dot{x}), \qquad A(p) \coloneqq A^c\bar{p}_c = \frac{m}{\ell}\,A_c\dot{x}^c \tag{B19}$$

. and to do this using equation (128), we need the following derivatives of the perturbation function:

$$\dot{\partial}_b h = m^{2r}\,s\,[A(p)]^{s-1}\,\dot{\partial}_b A(p) \tag{B20}$$

$$\delta_b h = m^{2r}\,s\,[A(p)]^{s-1}\,\delta_b A(p) \tag{B21}$$

$$\delta_a\dot{\partial}_b h = m^{2r}\Big[s\,[A(p)]^{s-1}\,\delta_a\dot{\partial}_b A(p) + s(s-1)\,[A(p)]^{s-2}\,(\delta_a A(p))(\dot{\partial}_b A(p))\Big]\,.$$

The expressions for the derivatives of $A(p)$, can be substituted from equation (135). After substitution these derivatives have the following expression:

$$\dot{\partial}_b h = m^{2r}\,s\,[A(\dot{x})]^{s-1}\,\frac{m}{\ell}\left(A_b - \frac{A(\dot{x})}{\ell^2}\dot{x}_b\right)$$

$$\delta_a h = m^{2r}\,s\,[A(\dot{x})]^{s-1}\,\frac{m}{\ell}\,\dot{x}^e\nabla_a A_e$$

$$\delta_a\dot{\partial}_b h = m^{2r+1}\,\frac{s}{\ell}\,[A(\dot{x})]^{s-1}\left[\partial_a A^c\left(g_{bc} - \frac{\dot{x}_b\dot{x}_c}{\ell^2}\right) + A^c\left(\Gamma_{abc} + \Gamma_{cab} - \frac{1}{\ell^2}\left(\dot{x}_d\Gamma^d{}_{ab}\dot{x}_c + \dot{x}_d\Gamma^d{}_{ca}\dot{x}_b\right)\right)\right]$$
$$+\, m^{2r+2}\,\frac{s(s-1)}{\ell^2}\,[A(\dot{x})]^{s-2}\,(\dot{x}^e\nabla_a A_e)\left(A_b - \frac{A(\dot{x})}{\ell^2}\dot{x}_b\right). \tag{B22}$$

Evaluating the following expression:

$$\dot{x}^a\delta_a\dot{\partial}_b h - \Gamma^c{}_{bj}\dot{x}^j\dot{\partial}_c h = m^{2r+1}\,\frac{s}{\ell}\,[A(\dot{x})]^{s-1}\left(\dot{x}^a\nabla_a A_b - \frac{1}{\ell^2}\,\dot{x}_b\dot{x}^a\dot{x}^e\nabla_a A_e\right)$$
$$+\, m^{2r+2}\,\frac{s(s-1)}{\ell^2}\,[A(\dot{x})]^{s-2}\left(\dot{x}^a\nabla_a A_e\dot{x}^e\right)\left(A_b - \frac{A(\dot{x})}{\ell^2}\,g_{bf}\dot{x}^f\right). \tag{B23}$$

Substituting everything in (128),

$$\ddot{x}^q + \Gamma^q{}_{ij}\dot{x}^i\dot{x}^j = \frac{\epsilon}{2m^2}m^{2r+1}\frac{s}{\ell}\,[A(\dot{x})]^{s-1}\Bigg\{g(\dot{x},\dot{x})\,g^{qb}\Big[\left(\dot{x}^a\nabla_a A_b - \frac{1}{\ell^2}\,\dot{x}_b\,\dot{x}^a\dot{x}^e\nabla_a A_e\right)$$
$$+\,\frac{m}{\ell}\,(s-1)[A(\dot{x})]^{-1}\left(\dot{x}^a\nabla_a A_e\dot{x}^e\right)\left(A_b - \frac{A(\dot{x})}{\ell^2}\,g_{bf}\dot{x}^f\right) - \dot{x}^e\nabla_b A_e\Big] + 2\,\dot{x}^q\dot{x}^a\dot{x}^e\nabla_a A_e\Bigg\}$$
$$= \frac{\epsilon}{2m^2}\,m^{2r+1}\frac{s}{\ell}\,[A(\dot{x})]^{s-1}\left\{\ell^2\,\dot{x}^a\left(\nabla_a A^q - \nabla^q A_a\right) \,+\, (m\ell)\,(s-1)\,[A(\dot{x})]^{-1}\,(\dot{x}^a\dot{x}^e\nabla_a A_e)\,A^q\right.$$
$$\left. +\,\left[\dot{x}^a\dot{x}^e\nabla_a A_e - (s-1)\,\frac{m}{\ell}\dot{x}^a\dot{x}^e\nabla_a A_e\right]\dot{x}^q\right\}. \tag{B24}$$

On first sight, this equation looks different compared to (91). However we can change the parametrization $\tau \to s(\tau)$ so that

$$\dot{x}(s(\tau)) = \frac{d}{d\tau}x(s(\tau)) = \frac{dx}{ds}\frac{ds}{d\tau} = x'\dot{s}\,,$$
$$\ddot{x}(s(\tau)) = x'\ddot{s} + x''\dot{s}^2\,, \tag{B25}$$

which turns the geodesic equation to

$$\begin{aligned} x''^q + \Gamma^q{}_{bc}x'^b x'^c = \frac{\epsilon}{2m^2}\, m^{2r+1}\frac{s}{\ell}\,[A(\dot{x}')]^{s-1}\Big\{&\ell^2\, x'^a\,(\nabla_a A^q - \nabla^q A_a) \\ &+ ((m\ell))(s-1)[A(x')]^{-1}\,(x'^a x'^e \nabla_a A_e)A^q\Big\} \\ &+ x'^q\left[\frac{\epsilon}{2m^2}\, m^{2r+1}\frac{s}{\ell}\,[A(\dot{x}')]^{s-1}\Big(x'^a x'^e\nabla_a A_e - (s-1)\,\frac{m}{\ell}(x'^a x'^e \nabla_a A_e) - \frac{\ddot{s}}{\dot{s}^2}\right] \end{aligned} \tag{B26}$$

The last term vanishes if one choose a parametrization of the curve such that

$$\frac{\ddot{s}}{\dot{s}^2} = \frac{\epsilon}{2m^2}\, m^{2r+1}\frac{s}{\ell}\,[A(x')]^{s-1}x'^a x'^e\nabla_a A_e\Big(1 - (s-1)\,\frac{m}{\ell}\Big)\,, \tag{B27}$$

which then turns the geodesic equation closer to (91) without the term multiplied by $\dot{x}^a$.

We see that the geodesic equation of GR is reobtained in both cases if

$$\dot{x}^a\big(\nabla_a A^q - \nabla^q A_a\big) \;+\; (m\ell)\,(s-1)\,[A(\dot{x})]^{-1}\,(\dot{x}^a\dot{x}^e\nabla_a A_e)\,A^q = 0 \tag{B28}$$

4. Deriving geodesic equations using the n-polynomial formula.

a. For $h = [A(p)]^3$

Plugging $n = 3$ into equation (147),

$$\begin{aligned} \mathrm{RHS} &= \frac{\epsilon}{2m^2}\Big(\ell^2 g^{qa}\big(3\nabla_b h_{acd} - \nabla_a h_{bcd}\big) - \dot{x}^q\dot{x}^a\nabla_a h_{bcd}\Big)\left(\frac{m}{\ell}\right)^3 \dot{x}^b\dot{x}^c\dot{x}^d \\ &= \frac{\epsilon m}{2\ell}\Big(\big[\,3\nabla_b h^q{}_{cd} - \nabla^q h_{bcd}\,\big]\,\dot{x}^b\dot{x}^c\dot{x}^d\Big) - \frac{\epsilon m}{2\ell^3}\,\dot{x}^q\dot{x}^a\nabla_a h_{bcd}\,\dot{x}^b\dot{x}^c\dot{x}^d, \end{aligned} \tag{B29}$$

Now, contracting the ps with the covariant derivative and replacing the ps with $\dot{x}$ terms

$$\begin{aligned}
\nabla_b h_{acd}\,\dot{x}^b\dot{x}^c\dot{x}^d &= (\nabla_b A_a)\,(A\cdot\dot{x})^2 + 2\,A_a\,(A\cdot\dot{x})\,\dot{x}^c\,\dot{x}^b\nabla_b A_c\\
\nabla_a h_{bcd}\,\dot{x}^b\dot{x}^c\dot{x}^d &= 3\,(A\cdot\dot{x})^2\,(\nabla_a A_k)\,\dot{x}^k,\\
\nabla^q h_{bcd}\,\dot{x}^b\dot{x}^c\dot{x}^d &= 3\,(A\cdot\dot{x})^2\,(\nabla^q A_k)\,\dot{x}^k.
\end{aligned}\tag{B30}$$

Therefore,

$$\begin{aligned}
\text{RHS} &= \frac{\epsilon}{2m^2}\Big(\ell^2 g^{qa}\big(3\nabla_b h_{acd} - \nabla_a h_{bcd}\big) - \dot{x}^q\dot{x}^a\nabla_a h_{bcd}\Big)\left(\frac{m}{\ell}\right)^3 \dot{x}^b\dot{x}^c\dot{x}^d\\
&= \frac{\epsilon m}{2\ell}\Big(\big(3(\nabla_b A^q)\,\dot{x}^b\,(A\cdot\dot{x})^2 + 6\,A^q\,(A\cdot\dot{x})\,\dot{x}^c\,\dot{x}^b\nabla_b A_c\\
&\quad - 3\,(A\cdot\dot{x})^2\,(\nabla^q A_k)\,\dot{x}^k\big) - \frac{3}{\ell^2}\,\dot{x}^q\dot{x}^a(A\cdot\dot{x})^2\,(\nabla_a A_k)\,\dot{x}^k\Big)\\
&= \frac{\epsilon m}{2\ell}\Big(3\,(A\cdot\dot{x})^2\,\dot{x}^k(\nabla_k A^q - \nabla^q A_k)\big)\; + 6\,A^q\,(A\cdot\dot{x})\,\dot{x}^c\,\dot{x}^b\nabla_b A_c - \frac{3}{\ell^2}\,\dot{x}^q\dot{x}^a(A\cdot\dot{x})^2\,(\nabla_a A_k)\,\dot{x}^k\Big)
\end{aligned}\tag{B31}$$

Here we can see that we get the exactly same expression using the n-polynomial formula as we had in (B10).

b. *For $h(x,p) = g(p,p)A(p)$*

Now, we start with

$$h(x,p) = g(p,p)A(p) = g^{ab}A^c p_a p_b p_c \Leftrightarrow h^{abc} = \frac{1}{3}(g^{ab}A^c + g^{bc}A^a + g^{ac}A^b) = g^{(ab}A^{c)}\,, \tag{B32}$$

Now specifying this specific form of h in the polynomial geodesic equation 147

$$h^{abc} = g^{ab}A^c, \qquad h_{abc} = g_{ai}g_{bj}g_{ck}h^{ijk} = g_{bc}A_a. \tag{B33}$$

Plugging $n = 3$ into equation 147, the RHS becomes,

$$\text{RHS} = \frac{\epsilon}{2m^2}\left(\ell^2 g^{qa}\Big(3\nabla_b h_{acd} - \nabla_a h_{bcd}\Big) - \dot{x}^q\dot{x}^a\nabla_a h_{bcd}\right)\bar{p}^b\bar{p}^c\bar{p}^d. \tag{B34}$$

Calculating the covariant derivatives:

$$\begin{aligned}
\nabla_b h_{acd} &= 2\,g_{ad}\nabla_b A_c + g_{cd}\nabla_b A_a,\\
\nabla_a h_{bcd} &= 3\,g_{cd}\nabla_a A_b.
\end{aligned}\tag{B35}$$

$$\mathrm{RHS} = \frac{\epsilon}{2m^2}\left(\ell^2 g^{qa}\Big(3\nabla_b h_{acd} - \nabla_a h_{bcd}\Big) - \dot{x}^q\dot{x}^a\nabla_a h_{bcd}\right)\bar{p}^b\bar{p}^c\bar{p}^d$$
$$= \frac{\epsilon}{2m^2}\Big[\, 6\,\nabla_b A_c\,\tfrac{m^3}{\ell}\dot{x}^b\dot{x}^c\dot{x}^q + 3\,\ell^2\,\nabla_b A^q\tfrac{m^3}{\ell}\dot{x}^b - 3\,\ell^2\nabla^q A_b\,\tfrac{m^3}{\ell}\dot{x}^b - 3\,\nabla_b A_c\,\tfrac{m^3}{\ell}\dot{x}^b\dot{x}^c\dot{x}^q\Big] \tag{B36}$$
$$= \frac{3\,\epsilon m}{2\ell}\Big[\ell^2\,\dot{x}^b\Big(\nabla_b A^q - \nabla^q A_b\Big) + \dot{x}^b\dot{x}^c\dot{x}^q\,\nabla_b A_c\Big] \tag{B37}$$

Here we can see that we get the exactly same expression using the n-polynomial formula as we had in (B18).

c. For Microscopic Recoil Model MDR

Let us start with the perturbation function,

$$h(x,p) = \Big(A^q(x)p_q\Big)^n. \tag{B38}$$

Where,

$$h_{b_1\ldots b_n} = A_{b_1}A_{b_2}\cdots A_{b_n}. \tag{B39}$$

Computing the necessary covariant derivatives,

$$\nabla_a h_{b_1\ldots b_n} = \sum_{j=1}^{n} A_{b_1}\cdots(\nabla_a A_{b_j})\cdots A_{b_n}.$$
$$\bar{\nabla}_{b_1} h_{ab_2\ldots b_n} = (\nabla_{b_1}A_a)A_{b_2}\ldots A_{b_n} + \sum_{j=2}^{n}(\nabla_{b_1}A_{b_j})A_a\prod_{\substack{k=2\\k\neq j}}^{n}A_{b_k}\Big].$$

Contracting with $\bar{p}^{b_1}\cdots\bar{p}^{b_n}$

$$\bar{p}^{b_1}\cdots\bar{p}^{b_n}\Big(n\,(\nabla_{b_1}A_a)A_{b_2}\cdots A_{b_n}\Big) = n(A\cdot\bar{p})^{n-1}\,\bar{p}^b\nabla_b A_a$$
$$= n\,\frac{m^n}{\ell^n}\,(A(\dot{x}))^{n-1}\,\dot{x}^b\nabla_b A_a\,,$$
$$\bar{p}^{b_1}\cdots\bar{p}^{b_n}\Big(n\sum_{j=2}^{n}(\nabla_{b_1}A_{b_j})A_a\prod_{k\neq j}A_{b_k}\Big) = n(n-1)\,(A\cdot\bar{p})^{n-2}\,A_a\,\bar{p}^b\bar{p}^c\,\nabla_b A_c$$
$$= n(n-1)\,\frac{m^n}{\ell^n}\,(A(\dot{x}))^{n-2}\,A_a\,\dot{x}^b\dot{x}^c\nabla_b A_c\,,$$
$$\bar{p}^{b_1}\cdots\bar{p}^{b_n}\Big(\nabla_a h_{b_1\ldots b_n}\Big) = n(A\cdot\bar{p})^{n-1}\,\bar{p}^b\nabla_a A_b = n\frac{m^n}{\ell^n}(A(\dot{x}))^{n-1}\dot{x}^b\nabla_a A_b\,,$$
$$(2-n)\,\dot{x}^q\dot{x}^a\bar{p}^{b_1}\cdots\bar{p}^{b_n}\,\nabla_a h_{b_1\ldots b_n} = n(2-n)\,\dot{x}^q\dot{x}^a(A\cdot\bar{p})^{n-1}\,\bar{p}^b\,\nabla_a A_b$$
$$= n(2-n)\,\frac{m^n}{\ell^n}(A(\dot{x}))^{n-1}\,\dot{x}^q\dot{x}^a\dot{x}^b\nabla_a A_b\,. \tag{B40}$$

Therefore,

$$
\begin{aligned}
\text{RHS} &= \frac{\epsilon}{2m^2}\Bigg(\ell^2 g^{qa}\Big(n\,\frac{m^n}{\ell^n}\,(A(\dot{x}))^{n-1}\,\dot{x}^b\nabla_b A_a + n(n-1)\,\frac{m^n}{\ell^n}\,(A(\dot{x}))^{n-2}\,A_a\,\dot{x}^b\dot{x}^c\nabla_b A_c \\
&\quad - \ell^2 g^{qa} n\frac{m^n}{\ell^n}(A(\dot{x}))^{n-1}\dot{x}^b\nabla_a A_b + n(2-n)\,\frac{m^n}{\ell^n}(A(\dot{x}))^{n-1}\,\dot{x}^q\dot{x}^a\dot{x}^b\nabla_a A_b\Bigg) \\
&= \frac{\epsilon}{2m^2}n\,\frac{m^n}{\ell^{n-2}}\,(A(\dot{x}))^{n-1}\,\dot{x}^b\Bigg(g^{qa}(\nabla_b A_a - \nabla_a A_b) + \frac{(n-1)}{A(\dot{x})}\,A^q\,\dot{x}^c\nabla_b A_c\Bigg) \\
&\quad + \frac{\epsilon}{2m^2}n(2-n)\,\frac{m^n}{\ell^n}(A(\dot{x}))^{n-1}\,\dot{x}^q\dot{x}^a\dot{x}^b\nabla_a A_b
\end{aligned}
\tag{B41}
$$

As we can see that we get a structurally similar equation to the one obtained in Hamiltonian framework (A21) except for the term that can be absorbed through reparameterization.

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