MASTER IN ASTROPHYSICS AND SPACE SCIENCE

Erasmus Mundus Master
in Astrophysics and Space Science

Master Thesis

Shadow of an Accelerated Black Hole in an Expanding Universe

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Contents

Abstract

This thesis investigates the shadow of a uniformly accelerated black hole embedded in an expanding de Sitter universe, modeled by the C-metric with a cosmological constant. We derive and analyze the null geodesics that determine the photon sphere and hence the shadow boundary. The shadow is computed for two natural classes of observers: static observers at fixed spatial coordinates and co-moving observers in the $\theta = \pi/2$ plane following the cosmological expansion. For static observers the shadow remains circular, while for co-moving observers its apparent size is modified by relativistic aberration and asymptotically approaches a finite value at large distances. These results extend previous studies of black hole shadows to spacetimes that incorporate both acceleration and cosmological expansion, providing a framework for interpreting observations of compact objects in realistic cosmological settings.

Introduction

Black holes have long been central objects of study in general relativity and astrophysics. Their extreme gravitational fields provide natural laboratories to probe the nature of spacetime, test Einstein's theory in the strong-field regime, and gain insights into the interplay between gravitation and cosmology. Among the many fascinating phenomena associated with black holes, one of the most striking is the black hole shadow — a dark region on the celestial sphere of an observer, corresponding to the capture of light by the event horizon. The first direct observation of a black hole shadow by the Event Horizon Telescope in 2019 has elevated the importance of theoretical studies of shadows, particularly in more general and physically realistic spacetimes.

While most early investigations of black hole shadows focused on stationary and asymptotically flat spacetimes such as the Schwarzschild and Kerr metrics, the Universe we inhabit is neither static nor asymptotically flat. The presence of a cosmological constant introduces large-scale expansion, and the possibility of black hole acceleration — arising from, for example, the interaction with cosmic strings or external fields — adds an additional layer of complexity. In this context, the C-metric provides an exact solution to Einstein's equations describing a uniformly accelerated black hole embedded in a de Sitter background. This spacetime captures both local strong-field features near the black hole and the global structure of an expanding universe.

The objective of this thesis is to investigate the shadow of an accelerated black hole in a cosmological setting, with particular emphasis on how the acceleration and cosmological constant affect the appearance of the shadow to different observers. We focus on two natural classes of observers: static observers, who remain at fixed spatial coordinates with respect to the black hole, and co-moving observers, who follow the cosmological expansion. The comparison between these perspectives provides physical insight into how local dynamics (black hole acceleration) and global structure (cosmological expansion) interplay in shaping observable features.

Our analysis proceeds as follows. We begin by introducing the C-metric and specifying its parameter and coordinate ranges, identifying the physically relevant region of spacetime between the inner and outer horizons. We then derive the equations governing lightlike geodesics, exploiting the Euler–Lagrange and Hamilton–Jacobi formalisms to obtain separable equations of motion. Special attention is paid to the existence of unstable photon orbits — generalized photon spheres — which determine the boundary of the black hole shadow.

For the static observer, we construct a local orthonormal tetrad, express the constants of motion of photons in this frame, and determine the angular radius of the shadow. The resulting shadow is found to be perfectly circular, with its size depending on the mass, acceleration, and cosmological constant of the black hole. Extending this to the co-moving observer requires a careful coordinate transformation that renders the metric into a cosmological form reminiscent of a Friedmann–Lemaître–Robertson–Walker (FLRW) spacetime. We do this for the co-moving observer on the plane $\theta = \pi/2$. The transformation allows us to define the co-moving observer's four-velocity, compute

their relative motion with respect to static observers, and apply the special-relativistic aberration formula to obtain the angular size of the shadow in this expanding frame.

The results of this work not only generalize previous studies of black hole shadows in asymptotically flat and de Sitter spacetimes but also provide a framework for analyzing shadows in more realistic, dynamical contexts. Such analyses are essential for interpreting future high-resolution observations of compact objects in cosmological environments.

This thesis is organized as follows. In Chapter 1, we introduce the C-metric describing a uniformly accelerated black hole in a de Sitter background, discuss its coordinate ranges, and identify the relevant physical region of the spacetime. Chapter 2 is devoted to the analysis of lightlike geodesics, where we derive the equations of motion using the Euler–Lagrange formalism and characterize the photon sphere. In Chapter 3, we construct the shadow as perceived by a static observer. Chapter 4 extends the analysis to a co-moving observer, for whom the spacetime is recast in a cosmological form. We adopt natural units with $c = 1$ and use metric signature $(-, +, +, +)$.

1 Metric

To investigate the shadow of an accelerated black hole, it is essential to analyze the behavior of lightlike geodesics in the spacetime geometry under consideration. In the present work, this geometry is described by the so-called $C-metric$, which represents an accelerated black hole embedded in a cosmological de Sitter background. The spacetime is characterized by three independent parameters: $m - mass$, $a - acceleration$, and $\Lambda - cosmological\ constant$. In what follows, we restrict attention to the case in which all parameters are non-negative. Additional constraints on their allowed ranges will be specified later. Before discussing these in detail, we introduce the C-metric, which arises as a special case of the most general Plebanski–Demianski solution. The particular form employed here follows the presentation given by Griffiths and Podolský (2009):

$$ds^2 = \frac{1}{\Omega^2(r,\theta)}\left[-\frac{Q(r)}{r^2}dt^2 + \frac{r^2}{Q(r)}dr^2 + \frac{r^2}{P(\theta)}d\theta^2 + P(\theta)r^2 sin^2\theta d\phi^2\right] \tag{1}$$

Where functions $\Omega(r,\theta), Q(r)$ and $P(\theta)$ are given by:

$$\Omega(r,\theta) = 1 - arcos\theta \tag{2}$$

$$Q(r) = (r^2 - 2mr)(1 - a^2r^2) - \frac{\Lambda}{3}r^4 \tag{3}$$

$$P(\theta) = 1 - 2amcos\theta \tag{4}$$

$$\tag{5}$$

It is worth noting that the given metric reduces to simpler form in the following three cases:

- $\Lambda = 0$ reduces to the metric that only depends on mass and acceleration parameters. Lightlike geodesics in this type of metric has been discussed by Frost and Perlick (2021). Even though we are working with a more complicated case, there are lots of similarities with this solution and we will be referencing it a lot in our work.

- $a = 0$ reduces to the well known Kottler metric. Lightlike geodesics in this metric has been studied extensively. Specificaly shadow calculatins has been done by Perlick et al. (2018).

Throughout this paper we will compare our results to the ones mentioned above to observe the physical role of Λ and a in the metric.
The metric [1] clearly shows that *curvature singularity* occurs at $r = 0$. The solutions of $Q(r) = 0$ are coordinate singularities which correspond to Killing horizons. The metric is not $assymptotically\ flat$. Before we move onto the coordinate ranges in this metric, we need to note that ∂_t and ∂_ϕ are the Killing vector fields. ∂_t is associated with the invariance of the metric with respect to time translations and ∂_ϕ is associated with the invariance of the metric with respect to rotations around the axis $sin\theta = 0$. The region where ∂_t is timelike and ∂r is spacelike is static. In the other regions causal structure of these two are reversed. In the following sections we will specify parameters and the region, where the conditions of ∂_t being timelike and $\partial_r, \partial_\theta, \partial_\phi$ being spacelike are satisfied.

1.1 Coordinate Ranges of the Metric

The t coordinate ranges over the entire set of real numbers, R. The angular coordinate ϕ, representing the azimuthal angle, ranges from 0 to 2π. The polar angle θ and the radial coordinate r require a more detailed analysis. If we start by examining the regularity of axis of symmetry at $\theta = 0$ and $\theta = \pi$ (this was done by Griffiths and Podolský (2009)), we see that the circumference over radius of the circle around each half-axis: at $\theta = 0$ and $\theta = \pi$ is $2\pi(1 \pm 2am)$ respectively. This implies the existence of a conical singularity with a different conicity (unless $am = 0$). The conical singularity with constant deficit angle along the half axis $\theta = \pi$ can be interpreted as representing a semi-infinite cosmic string under tension. This extends from the source at $r = 0$ right out to conformal infinity. This naturally leads to the interpretation that the metric [1] represents a Schwarzschild-like black hole that is being accelerated along the axis $\theta = \pi$ in a de Sitter space by the action of a force which corresponds to the tension in a cosmic string.
For the range of the θ coordinate $P(\theta)$ must be positive. If $2ma > 1$ then this bounds the θ coordinate away from the axis $sin\theta = 0$ which seems to be non physical, so we will restrict values of the acceleration parameter in this way:

$$2ma < 1 \tag{6}$$

In order to discuss the range of the θ coordinate in more detail and the range of r, we have to study functions in [2] and [3] more in detail. More specifically, we need to study their zeros.

First we discuss the solution of the equation:

$$\Omega(r, \theta) = 0 \tag{7}$$

It is obvious that according to [1] and [7], coordinates where the following equality is satisfied, corresponds to conformal infinity.

$$1 - ar\cos\theta = 0 \tag{8}$$

Since we are only considering the case of positive radial coordinates and positive acceleration parameter, this singularity only exists on the northern hemisphere ($\cos\theta > 0$). It starts at $r = 1/a$ at the pole and continues to the right - reaching $r \to \infty$ on the $\theta = \pi/2$ plane. On the southern hemisphere conformal infinity does not exist. For the sake of physical relevance we should avoid the points where $r = \frac{1}{a\cos\theta}$. This will obviously restrict coordinate ranges on the northern hemisphere.

Last but not least, we need to study zeros of the function $Q(r)$ for the coordinates $r > 0$. We will first look at the function $g(r) = \frac{Q(r)}{r}$

$$g(r) = -(a^2 + \frac{\Lambda}{3})r^3 + 2ma^2r^2 + r - 2m \tag{9}$$

By the Descarte's rule University (nd), we see that this equation can have either 2 positive roots or 0 positive roots. Since we are interested in the restrictions of the

range of radial coordinate, we will focus on the case of two positive zeros. Restrictions that this sets on the parameters m, a, Λ will be discussed in the next section. But meanwhile it is easier to study these two zeros by dealing with a function:

$$\frac{Q(r)}{r^4} = \left(1 - \frac{2m}{r}\right)\left(\frac{1}{r^2} - a^2\right) - \frac{\Lambda}{3} \tag{10}$$

If we consider the case of Λ being very small, the zeros of the above mentioned function are very close to $r_1 = 2m$ and $r_2 = \frac{1}{a}$. Because of the extra Λ term in the function, we have that the horizons of the black hole described with $g(r) = 0$ are r_{bh} and r_a, where:

$$r_{bh} > 2m \tag{11}$$

$$r_a < \frac{1}{a} \tag{12}$$

Hence the range of radial coordinate where ∂_t is timelike and ∂_r is spacelike is $r \in (r_{bh}, r_a)$. In this work we are not going to attempt to calculate exact values of r_{bh} or r_a

1.2 Parameter Ranges for the Metric

We already know that we are considering a metric with positive parameter m, a, Λ. Also while discussing the coordinate ranges , we have already mentioned one restriction for the parameters:

$$2ma < 1 \tag{13}$$

Let's discuss one more restriction that is based on the existence of horizons in the metric. For this, we should study the $g(r)$ function [9] in more detail:

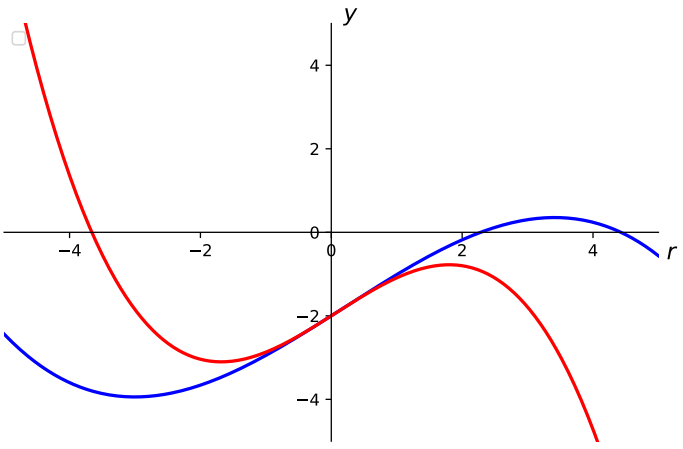

Figure 1: This plot describes $g(r)$ for two different set of parameters $(m, a, (\Lambda/3)^2)$: For the blue and red curve we have $(1, 0.1, 0.15)$ and $(1, 0.1, 0.01)$ respectively.

1. $g(0) = -2m$: graph of $g(r)$ has to cross the $y - axis$ as in Fig.1 blue curve.

2. $g'(r) = 0$ has two solutions:

$$r_3 = \frac{4ma^2 - \sqrt{16m^2a^4 + 4(3a^2 + \Lambda)}}{2(3a^2 + \Lambda)} < 0 \tag{14}$$

$$r_4 = \frac{4ma^2 + \sqrt{16m^2a^4 + 4(3a^2 + \Lambda)}}{2(3a^2 + \Lambda)} > 0 \tag{15}$$

3. $g''(r_3) > 0$ and $g''(r_4) < 0$, which means that when $r > 0$, specifically at $r_{bh} < r < r_a$ the graph of $g(r)$ is concave downwards.

In order $g(r)$ and hence $Q(r)$ to have at least one positive zero, the metric parameters m, a, Λ need to satisfy following restriction:

$$f(r_4) \geq 0 \tag{16}$$

- For $m > 0, a > 0, \Lambda > 0$ we get: $0 < m < m_3$ or $m > m_4$.
  Here m_3 and m_4 are largest and the second largest roots of the polynomial:

$$h(x) = 48a^6x^4 - (24a^4 + 36a^2\Lambda + 9\Lambda^2) + 3a^2 + \Lambda \tag{17}$$

  Which are given as:

$$m_3 = \frac{\sqrt{24a^4 + 36a^2\Lambda + 9\Lambda^2 - \sqrt{3\Lambda(8a^2 + 3\Lambda)^3}}}{4\sqrt{6}a^3} \tag{18}$$

$$m_4 = \frac{\sqrt{24a^4 + 36a^2\Lambda + 9\Lambda^2 + \sqrt{3\Lambda(8a^2 + 3\Lambda)^3}}}{4\sqrt{6}a^3} \tag{19}$$

  We can easily see that $m_4 > 1/(2a)$. Since we have already restricted our parameter range with [13], this means that [36] does not set any extra restriction on the range of m. On the other hand, inequality $0 < m < m_3$ gives non-trivial restriction on the parameters and needs to be taken into account with [13].

- On the other hand if $m > 0, \Lambda > 0, a = 0$, the inequality reduces to:

$$0 < m < \sqrt{\frac{1}{9\Lambda}} \tag{20}$$

  Which is exactly the range for m that was suggested by Perlick et al. (2018).

Since we know the restrictions on the parameter ranges and coordinate ranges, we are ready to move on to studying the equations of motion of light in the vicinity of the C-metric.

2 Lightlike Geodesics

Since the objective of this work is to investigate the shadow of an accelerated black hole, our primary focus will be on the propagation of light in the corresponding spacetime geometry. To this end, it is necessary to derive the equations governing lightlike geodesics. This can be accomplished using the Euler–Lagrange formalism. The Lagrangian for the system is given by:

$$L = \frac{1}{2\Omega^2(r,\theta)}\left[-\frac{Q(r)}{r^2}\dot{t}^2 + \frac{r^2}{Q(r)}\dot{r}^2 + \frac{r^2}{P(\theta)}\dot{\theta}^2 + P(\theta)(\sin^2\theta)r^2\dot{\phi}^2\right] \tag{21}$$

The momenta, which is conjugate of the four coordinates is given by $\frac{\partial L}{\partial \dot{x^\nu}} = p_\nu$. Using this, we get t and ϕ components of four-momenta.

$$p_t = -\frac{Q(r)}{r^2\Omega^2(r,t)}\dot{t} \tag{22}$$

$$p_\phi = \frac{P(\theta)(\sin^2\theta)r^2}{\Omega^2(r,\theta)}\dot{\phi} \tag{23}$$

Using Euler-Lagrange equation: $\frac{d}{ds}p_\nu = \frac{\partial L}{\partial x^\nu}$, we can derive the above-mentioned components of the lightlike geodesics.

$$\frac{d}{ds}p_t = \frac{\partial L}{\partial t} = 0 \tag{24}$$

$$\frac{d}{ds}p_\phi = \frac{\partial L}{\partial \phi} = 0 \tag{25}$$

As we see here, due to the fact that we have translation symmetries with respect to t and ϕ coordinates, p_t and p_ϕ are conserved quantities throughout the motion and we denote them with $-E$ and L_z respectively. This is because temporal translation is usually associated with the energy conservation (energy denoted by E), on the other hand, angular symmetry is associated to angular momentum conservation (angular momentum denoted by L_z). So finally we have the following equations:

$$-\frac{Q(r)}{r^2\Omega^2(r,t)}\dot{t} = -E \tag{26}$$

$$\frac{P(\theta)(\sin^2\theta)r^2}{\Omega^2(r,\theta)}\dot{\phi} = L_z \tag{27}$$

We can follow the same principle for the r and θ components of the equations, but since we have no obvious symmetries here the equations turn out to be complicated, so we are going to switch to the Hamiltonian formalism in order to get simpler forms for these components.

2.1 Separation of Equations of Motion

As we have already mentioned, canonical momenta is given by:

$$p_\nu = \frac{\partial L}{\partial x^\nu} = g_{\mu\nu}\dot{x}^\mu \tag{28}$$

If we multiply both sides with $g^{\sigma\nu}$, we get:

$$g^{\sigma\nu}p_\nu = g^{\sigma\nu}g_{\nu\mu}\dot{x}^\mu \tag{29}$$

$$\dot{x}^\sigma = g^{\sigma\nu}p_\nu \tag{30}$$

Hamiltonian is given by:

$$H(x,p) = \frac{1}{2}g^{\mu\nu}p_\mu p_\nu \tag{31}$$

$$H = L = -\frac{1}{2}\epsilon \tag{32}$$

For lightlike geodesics we have $\epsilon = 0$ and:

$$L = H = 0 \tag{33}$$

Which gives:

$$H\left(x, \frac{\partial S}{\partial x}\right) = 0 \tag{34}$$

Here we have replaced the argument of the Hamiltonian p_ν by $\frac{\partial S}{\partial x^\nu}$. Our goal is to find a complete integral of this partial differential equation. The usual trick to find such solution is with separation ansatz and that is what we are going to use:

$$S(t,r,\theta,\phi) = S_t(t) + S_r(r) + S_\theta(\theta) + S_\phi(\phi) \tag{35}$$

We can put this in more simplified form and have:

$$S(t,r,\theta,\phi) = -Et + S_r(r) + S_\theta(\theta) + L_z\phi \tag{36}$$

Next step is to plug [36] into $H = 0$, but for this we need to find inverse of $g_{\mu\nu}$, which can be easily calculated:

$$g^{-1} = \Omega(r,\theta)^2 \begin{pmatrix} -\frac{r^2}{Q(r)} & 0 & 0 & 0 \\ 0 & \frac{Q(r)}{r^2} & 0 & 0 \\ 0 & 0 & \frac{P(\theta)}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{P(\theta)(\sin^2\theta)r^2} \end{pmatrix} \tag{37}$$

So our Hamiltonian is the following:

$$H = \frac{1}{2}g^{\mu\nu}p_\mu p_\nu = \frac{1}{2}\Omega(r,\theta)^2\left[-\frac{r^2}{Q(r)}p_t^2 + \frac{Q(r)}{r^2}p_r^2 + \frac{P(\theta)}{r^2}p_\theta^2 + \frac{1}{P(\theta)\sin\theta^2 r^2}p_\phi^2\right] = 0 \tag{38}$$

$$\frac{1}{2}\Omega(r,\theta)^2\left[-\frac{r^2}{Q(r)}E^2 + \frac{Q(r)}{r^2}p_r^2 + \frac{P(\theta)}{r^2}p_\theta^2 + \frac{1}{P(\theta)(\sin^2\theta)r^2}L_z^2\right] = 0 \tag{39}$$

In order to solve [39], we need to remember that in the region of outer communication $\Omega(r,\theta)$ is never 0, which means that our final equation reduces to:

$$-\frac{r^2}{Q(r)}E^2+\frac{Q(r)}{r^2}\left(\frac{\partial S_r}{\partial r}\right)^2+\frac{P(\theta)}{r^2}\left(\frac{\partial S_\theta}{\partial \theta}\right)^2+\frac{1}{P(\theta)\sin^2\theta r^2}L_z^2=0 \tag{40}$$

At this point we can separate [40] into two equations:

$$-\left(\frac{\partial S_r}{\partial r}\right)^2 Q(r)+\frac{E^2r^4}{Q(r)}=K \tag{41}$$

$$\left(\frac{\partial S_\theta}{\partial \theta}\right)^2 P(\theta)+\frac{L_z^2}{P(\theta)\sin^2\theta}=K \tag{42}$$

Here, K is a separation constant. It is often referred to in the literature as a Carter-like constant, due to the analogy with the separation of the geodesic equations in the Kerr spacetime, where a similar procedure leads to the Carter constant.
The Equations in [41-42] lead us to the following:

$$p_r^2=\left(\frac{\partial S_r}{\partial r}\right)^2=\frac{-K}{Q(r)}+\frac{E^2r^4}{Q(r)^2} \tag{43}$$

$$p_\theta^2=\left(\frac{\partial S_\theta}{\partial \theta}\right)^2=\frac{K}{P(\theta)}-\frac{L_z^2}{P(\theta)^2\sin^2\theta} \tag{44}$$

And going back to [28] we have:

$$p_r=\frac{\partial L}{\partial \dot{r}}=\frac{r^2\dot{r}}{Q(r)\Omega^2(r,\theta)} \tag{45}$$

$$p_\theta=\frac{\partial L}{\partial \dot{\theta}}=\frac{r^2\dot{\theta}}{P(\theta)\Omega^2(r,\theta)} \tag{46}$$

Finally, combining [43-46], we get r and θ components of the lightlike geodesics:

$$\frac{r^2\dot{r}}{\Omega^2(r,\theta)}=\pm\sqrt{-KQ(r)+E^2r^4} \tag{47}$$

$$\frac{r^2\dot{\theta}}{\Omega^2(r,\theta)}=\pm\sqrt{KP(\theta)-\frac{L_z^2}{sin\theta^2}} \tag{48}$$

2.2 Types of Lightlike Geodesics

In order to study the types of lightlike geodesics, we will follow the analysis of the lightlike geodesic equations[26-27] and [47-48]. Analysis becomes more simple if we write down the equations in terms of Mino-like parameter Mino (2003). The parameter will be given according to the following formula:

$$\frac{d\lambda}{ds}=\frac{\Omega(r,\theta)^2}{r^2} \tag{49}$$

Here λ denotes the Mino-like parameter. So now the lightlike geodesic equations will have the following form:

$$\frac{dt}{d\lambda} = \frac{Er^4}{Q(r)} \tag{50}$$

$$\frac{d\phi}{d\lambda} = \frac{L_z}{P(\theta)\sin^2\theta} \tag{51}$$

$$\left(\frac{dr}{d\lambda}\right)^2 = \rho(r) \tag{52}$$

$$\left(\frac{d\theta}{d\lambda}\right)^2 = \Theta(\theta) \tag{53}$$

Here functions $\rho(r)$ and $\Theta(\theta)$ are given by:

$$R(r) = E^2r^4 - KQ(r) \tag{54}$$

$$\Theta(\theta) = KP(\theta) - \frac{L_z^2}{\sin\theta^2} \tag{55}$$

It is worth noting that if $\Lambda = 0$, our equations are completely same as the lightlike geodesic equations in Frost and Perlick (2021) as they should be.

2.2.1 θ Motion

In order to analyze lightlike geodesics, we first look at the θ motion, which is completely described with [53] and [55]. This equation is identical to the one that authors analyze in Frost and Perlick (2021). We just summarize the results here.
The motion in θ component becomes very easy to track if we rewrite the equation in the following form:

$$\frac{\sin\theta^2}{K}\left(\frac{d\theta}{d\lambda}\right)^2 + V_\theta(\theta) = -\frac{L_z^2}{K} \tag{56}$$

$$V_\theta(\theta) = -P(\theta)\sin^2\theta \tag{57}$$

Here we have to be careful with the constant K. According to the [42] $K \geq 0$. We will discuss $K = 0$ case at the end of this subsection, but for now, we consider $K > 0$, so it is possible to divide the initial equation by K and get [56]. This formula also shows that the motion is only possible when $V_\theta(\theta) < -L_z^2/K$. If we look at the Fig. 2 of $V_\theta(\theta)$ below, we can see that the function reaches it's minimum at $\theta = \theta_{ph}$, where θ_{ph} is given by:

$$\cos\theta_{ph} = \frac{-2am}{1+\sqrt{1+12a^2m^2}} \tag{58}$$

Given this information, we can have three types of motion in θ component of the null geodesics:

- $K = \dfrac{L_z^2}{\sin\theta_{ph}{}^2 P(\theta_{ph})} \neq 0$: In this case the null geodesics remain at $\theta = \theta_{ph}$. This is why the authors denoted this value of θ by θ_{ph} - it represents a *photon*

cone. Lightlike geodesics that have this specific relation between their constants of motion, remain in this cone. It is worth noting that as $\cos\theta_{ph} < 0$, this cone is on the southern hemisphere of the black hole.

- $K \neq \dfrac{L_z^2}{\sin\theta_{ph}^2 P(\theta_{ph})}$: In this case the null geodesics oscillate between θ_{min} and θ_{max}, where $0 < \theta_{min} < \theta_{ph}$ and $\theta_{ph} < \theta_{max} < \pi$. Interesting thing to note here is that these geodesics start in the range $0 < \theta_{min} < \theta_{ph}$, they can cross the $\theta = \pi/2$ plane, but if they start in the range $\theta_{ph} < \theta_{max} < \pi$, they can never cross it and have to remain in the southern hemisphere.

- $K = 0$: In this case, according to the [53] and [55], $L_z = 0$. Which means that $\dfrac{d\theta}{d\lambda} = 0$ and $\dfrac{d\phi}{d\lambda} = 0$. These type of geodesics are called principal null geodesics. They only have radial components and will be discussed in the next subsection.

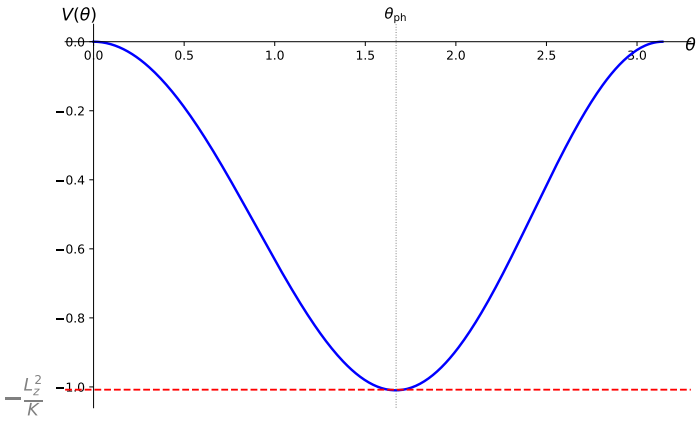

Figure 2: This plot describes effective potential $V_\theta(\theta)$ of the lightlike motion in θ component.

2.2.2 r Motion

In order to describe the r-motion of the light, we need to analyze [52].

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 r^4 - KQ(r) \tag{59}$$

- If $K = 0$, we get principal null geodesic, which were already mentioned before, in the previous section. The geodesic equation takes the form:

$$\frac{dr}{d\lambda} = \pm E r^2 \tag{60}$$

Here the plus sign denotes the outgoing and the minus sign denotes the ingoing null geodesics. The solution of the following problem will be given by:

$$r(\lambda) = \frac{r_0}{1 \pm r_0 E(\lambda - \lambda_0)} \tag{61}$$

Where λ_0 denotes the parameter value where the motion starts and $r_0 = r(\lambda_0)$ - is the initial condition.

Now we can focus on the case when $K > 0$ and write [59] in the following form:

$$\frac{1}{Kr^4}\left(\frac{dr}{d\lambda}\right)^2 + V_r(r) = \frac{E^2}{K} \tag{62}$$

$$V_r(r) = \frac{Q(r)}{r^4} = \left(1 - \frac{2m}{r}\right)\left(\frac{1}{r^2} - a^2\right) - \frac{1}{3}\Lambda \tag{63}$$

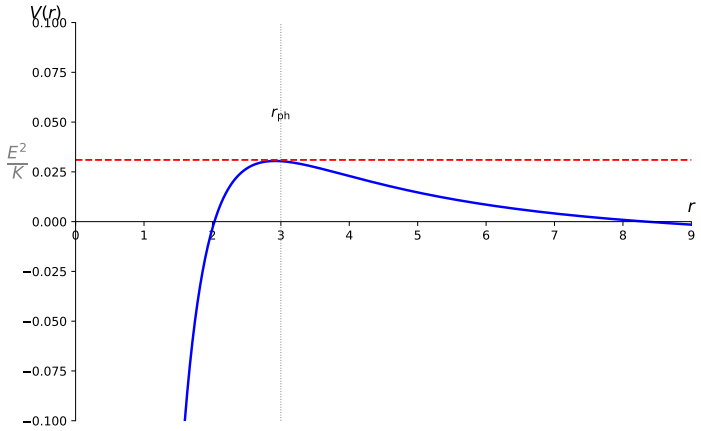

Figure 3: This plot describes effective potential $V_r(r)$ of the lightlike motion in r component.

This form of the equation is especially convenient to classify different types of geodesic equations. First, we note that $V_r(r) > 0$ as we are interested in the geodesics that move through the area of outer communication, which means that $Q(r) > 0$. Once we graph the potential function $V_r(r)$, we clearly see what types of geodesics are expected.
Equation [62] suggests that the geodesics will exist if $Q(r)/r^4 < E^2/K$. The maximum of the function $V_r(r)$ is reached at r_{ph}, where:

$$\frac{d}{dr}\left(\frac{Q(r)}{r^4}\right)|_{r_{ph}} = 0 \tag{64}$$

$$r_{ph} = \frac{6m}{1 + \sqrt{1 + 12a^2m^2}} \tag{65}$$

Now three other types of r motion can be identified:

- If $0 < \frac{E^2}{K} < \frac{Q(r_{ph})}{r_{ph}^2}$: In this case null geodesics have no turning points, which means that light that goes towards the center of the black hole will fall into it and will not be able to escape.

- $\frac{E^2}{K} > \frac{Q(r_{ph})}{r_{ph}^2}$: In this case if light can either be in the domain $r_{bh} < r < r_{max}$, where $r_{max} < r_{ph}$, or in the domain $r_{min} < r < r_a$, where $r_{min} > r_{bh}$.

- If $\frac{E^2}{K} = \frac{Q(r_{ph})}{r_{ph}^4}$: In this case the null geodesics on the photon sphere are asymptotically approaching the photon sphere in the future or in the past, hence notion of r_{ph}. This specific case will be the most relevant for us since it determines the surface that creates the contour of the black hole shadow.

3 Shadow of an Accelerated Black Hole Seen by a Static Observer

In general relativity, the shadow of a black hole corresponds to the set of directions from which light rays, when traced backwards in time from the observer's location, asymptotically approach unstable photon orbits instead of reaching distant light sources. The apparent boundary of the shadow is therefore determined by those lightlike geodesics that separate trajectories falling into the black hole from those that escape to infinity. The formation of the shadow is well illustrated in Perlick et al. (2018). Here we show the same simulation on Fig.5. In the preceding chapter, in [65] more specifically, we identified the photon sphere of the accelerated black hole metric.

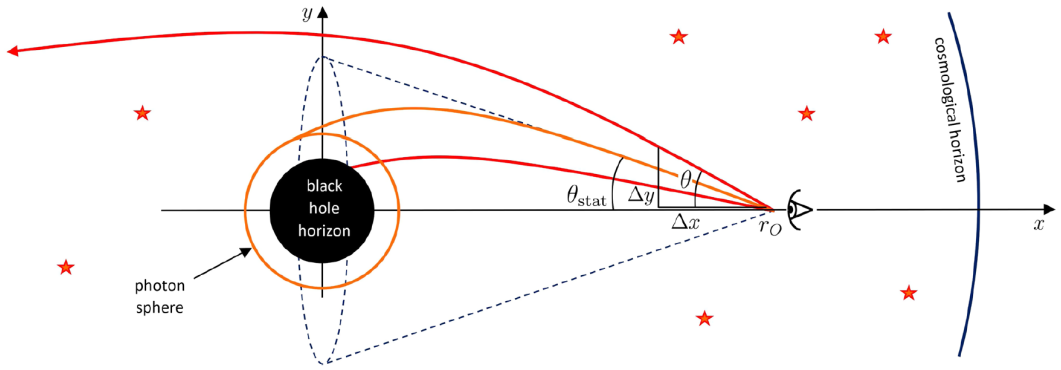

Figure 4: Formation of the shadow as seen by a static observer in the Kottler spacetime,Perlick et al. (2018)

In order to calculate angular radius of the shadow we will be following the usual astronomical convention. We use latitude-longitude coordinates on the celestial sphere of an observer. For this purpose we first fix a static observer at the coordinates $x^\nu = (t_O, r_O, \theta_O, \phi_O)$ between two horizons r_{bh} and r_a. Because of the symmetry of the spacetime, it is irrelevant which values we choose for t_O and ϕ_O. At the position of the observer we now introduce an orthonormal tetrad following Grenzebach Grenzebach et al. (2015) and Grenzebach (2016).

$$e_0 = \frac{\Omega(r,\theta)r}{\sqrt{Q(r)}}\partial_t|_{x^\nu_O} \tag{66}$$

$$e_1 = \frac{\Omega(r,\theta)\sqrt{P(\theta)}}{r}\partial_\theta|_{x^\nu_O} \tag{67}$$

$$e_2 = -\frac{\Omega(r,\theta)}{r\sin\theta\sqrt{P(\theta)}}\partial_\phi|_{x^\nu_O} \tag{68}$$

$$e_3 = \frac{-\Omega(r,\theta)\sqrt{Q(r)}}{r}\partial_r|_{x^\nu_O} \tag{69}$$

For the static observer - moving along the t-line, this tetrad also remains static. This set of orthonormal vectors are especially handy because in observer's frame of reference metric becomes locally flat, which allows the observer to define directions, angles, and

a celestial sphere as if they were in special relativity. So tangent vector of any curve can be expressed this way:

$$\frac{d\eta}{d\lambda} = \sigma(-e_0 + sin\Sigma\cos\psi e_1 + \sin\Sigma\sin\psi e_2 + \cos\Sigma e_3) \tag{70}$$

Here σ is a normalization constant.

$$\sigma = g\left(\frac{d\eta}{d\lambda}, e_0\right) = g(-e_0, e_0) \tag{71}$$

In general any tangent vector of a curve $\eta(\lambda)$ can be written in the following way:

$$\frac{d\eta}{d\lambda} = \frac{dt}{d\lambda}\partial_t + \frac{dr}{d\lambda}\partial_r + \frac{d\theta}{d\lambda}\partial_\theta + \frac{d\phi}{d\lambda}\partial_\phi \tag{72}$$

For the light like geodesics $\frac{dt}{d\lambda}, \frac{dr}{d\lambda}, \frac{d\theta}{d\lambda}, \frac{d\phi}{d\lambda}$ are given by [50-53]. Plugging them into [72] and comparing it to [70], we can find:

$$E = \frac{\sqrt{Q(r_O)}}{r_O\Omega(r_O, \theta_O)} \tag{73}$$

$$L_z = \frac{r_O\sqrt{P(\theta_O)}\sin\theta_O\sin\Sigma\sin\psi}{\Omega(r_O, \theta_O)} \tag{74}$$

$$K = \frac{r_O^2\sin^2\Sigma}{\Omega^2(r_O, \theta_O)} \tag{75}$$

These integrals of motion describe null geodesics that pass at x_O^ν, hence ones that are seen by the observer. Now in order to calculate the contour of the shadow of a black hole, we need these conserved quantities to satisfy equations that restrict them to lightlike geodesics that pass through the photon sphere: $r = r_{ph}$. This was derived in the previous section and is given by:

$$\frac{E^2}{K} = \frac{Q(r_{ph})}{r_{ph}^4} \tag{76}$$

So at the end using [73],[75],[76], we get:

$$\sin\Sigma_{ph} = \frac{\sqrt{Q(r_O)}}{\sqrt{Q(r_{ph})}}\frac{r_{ph}^2}{r_O^2} \tag{77}$$

It is easy to notice from [70], that Σ represents an angular diameter of the shadow. As the result shows, the final formula does not depend on ψ, which means that the shadow is circular. We have calculated the shadow for a static observer, which means we have assumed that the 4-velocity of the observer is parallel to ∂_t. From this result the shadow for any moving observer follows immediately by applying, on the tangent space at each point, the special-relativistic aberration formula. As the aberration formula maps circles onto circles, the shadow is circular for every observer. Our final formula involves cases when $a = 0$ and $\Lambda = 0$ and completely matches the results obtained in Perlick et al. (2018) and Frost and Perlick (2021) respectively.

4 Shadow of an Accelerated Black Hole Seen by a Co-moving Observer

Now that we have found the angular size of the shadow of the black hole for a static observer, we are ready to introduce a so-called co-moving observer. The main objective of this paper is to calculate the angular size of the shadow for the co-moving observer. In contrast to the static observer, who remains at a fixed spatial coordinate with respect to the black hole, a co-moving observer is defined with respect to the cosmological expansion. Such an observer follows the Hubble flow and thus moves radially outward in accordance with the expansion driven by the cosmological constant. Physically, this corresponds to an observer whose motion is adapted to the large-scale structure of the universe rather than anchored to the black hole.
In what follows, we will first establish the four-velocity of the co-moving observer, then find it's velocity with respect to the static observer and finish with calculating the shadow size by using aberration formula given bellow:

$$\sin^2\Sigma_c = \left(1-v^2\right)\frac{\sin^2\Sigma_{ph}}{\left(1-v\cos\Sigma_{ph}\right)^2} \tag{78}$$

Here Σ_c represents the angular size of the black hole shadow as seen by a co-moving observer and v is relative velocity of the co-moving observer with respect to the static observer. It is important to note that $\cos\Sigma_{ph}$ will be determined with the help of formula: $\cos\Sigma_{ph} = \pm\sqrt{1-\sin^2\Sigma_{ph}}$. Here the sign depends on the radial coordinate of the observer: We know that Σ_{ph} lies between $\pi/2$ and π for $r_{bh} < r < r_{ph}$ and that it lies between 0 and $\pi/2$ for $r > r_{ph}$. Therefore, we rewrite [78] as:

$$\sin^2\Sigma_c = \left(1-v^2\right)\frac{\sin^2\Sigma_{ph}}{\left(1+v\sqrt{1-\sin^2\Sigma_{ph}}\right)^2} \quad when \quad r_{bh} < r < r_{ph} \tag{79}$$

$$\sin^2\Sigma_c = \left(1-v^2\right)\frac{\sin^2\Sigma_{ph}}{\left(1-v\sqrt{1-\sin^2\Sigma_{ph}}\right)^2} \quad when \quad r > r_{ph} \tag{80}$$

This method was introduced in Perlick et al. (2018). There authors find appropriate coordinate transformation $(t, r, \theta, \phi) \to (\tilde{t}, \tilde{r}, \theta, \phi)$, which brings the metric into a form reminiscent of cosmological models. In particular, by redefining the temporal and radial coordinates, the line element can be expressed in a diagonal form analogous to that of a Friedmann–Lemaître–Robertson–Walker (FLRW) universe. Our goal is to find coordinate transformation: $(t, r, \theta, \phi) \to (T, R, \theta, \phi)$, which gives the metric following form:

$$ds^2 = -dT^2 + e^{kT}(dR^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2)) \tag{81}$$

This way an observer that moves along $\frac{\partial}{\partial T}$ curve sees the universe expanding exponentially with the expansion rate k. In this research we will do this for observers on the

$\theta = \pi/2$ plane of the black hole that move in a way that their polar coordinate remains constant: $\theta = \pi/2$. So our initial metric has the form:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\phi^2 \tag{82}$$

$$f(r) = 1 - \frac{2m}{r} + 2a^2 mr - A^2 r^2 \tag{83}$$

$$A^2 = \frac{\Lambda}{3} + a^2 \tag{84}$$

And we want to find a coordinate transformation that transforms the initial metric in the following form:

$$ds^2 = -dT^2 + e^{kT}(dR^2 + R^2 d\phi^2) \tag{85}$$

4.1 Coordinate Transformation: $(t, r, \theta, \phi) \to (T, R, \theta, \phi)$

We will use the coordinate transformation that is described by:

$$r = 2m + Re^{AT}\left(1 - \frac{m}{2Re^{AT}}\left(1 - \frac{a^2}{A^2}\right)\right)^2 \tag{86}$$

$$t = T + \int_{w_0}^{Re^{AT}} \frac{A\left(2m + w\left(1 - h(w)\right)^2\right)^2 (1 + h(w))^2\, dw}{w\left((1 - h(w))^2 - A^2\left(2m + w(1 - h(w))^2\right)^2 (1 + h(w))^2\right)} \tag{87}$$

with $h(w) = \frac{m}{2w}\left(1 - \frac{a^2}{A^2}\right)$.
Transformation of the coordinate differentials shows that:

$$dT = dt - \frac{\sqrt{A^2 r^2 - 2ma^2 r}dr}{\sqrt{1 - \frac{2m}{r}}f(r)} \tag{88}$$

$$\frac{dR}{AR} = -dt + \frac{1 - \frac{2m}{r}dr}{\sqrt{A^2 r^2 - 2ma^2 r}f(r)} \tag{89}$$

Plugging the differentials into the initial metric, we get:

$$g_{TT} = -\left(1 - \frac{2m}{r}\right) \tag{90}$$

$$g_{RT} = g_{TR} = 0 \tag{91}$$

$$A^2 R^2 g_{RR} = A^2 r^2 - 2ma^2 r \tag{92}$$

So the vector $\partial/\partial T$ is timelike for $2m < r < \infty$. The spatial part of the metric is of the form:

$$g_{RR}dR^2 + r^2 d\phi^2 = e^{2AT}\left(\sigma dR^2 + \rho R^2 d\phi^2\right) \tag{93}$$

Where σ and ρ go to 1 for $T \to \infty$. So finally, after long time has passed $T \to \infty$, we get exactly [85]. Which means that the observer that moves along the $\partial/\partial T$ can be defined as a co-moving observer. We will now find its 4-velocity. For this we need:

$$\frac{\partial}{\partial T} = \frac{\partial t}{\partial T}(t,r)\frac{\partial}{\partial t} + \frac{\partial r}{\partial T}(t,r)\frac{\partial}{\partial r} \tag{94}$$

In order to define the 4-velocity in terms of the original coordinates. Partial derivatives in [94] can be expressed by writing the equivalent transformation to [88-89] in terms of the Gaussian basis vector fields:

$$\frac{\partial}{\partial T} = \frac{1-\frac{2m}{r}}{f(r)}\frac{\partial}{\partial t} + \sqrt{1-\frac{2m}{r}}\sqrt{A^2r^2 - 2ma^2r}\frac{\partial}{\partial r} \tag{95}$$

On the other hand, 4-velocity of the static observer is along $\partial/\partial t$.

4.2 Relative Velocity of the Co-moving Observer with Respect to the Static Observer

In order to find the final unknown term in [79-80], we need to find the relative velocity of the co-moving observer with respect to the static observer. For this, we will use [95] from the previous section, the fact that static observer moves along $\partial/\partial t$ and the special-relativistic equation:

$$g_{\mu\nu}U_s^\mu U_c^\nu = \frac{-1}{\sqrt{1-v^2}} \tag{96}$$

where $U_s^\mu \partial/\partial x^\mu$ is the four-velocity vector of the static observer and $U_c^\mu \partial/\partial x^\mu$ is the four-velocity of the co-moving observer.

$$U_s^\mu \frac{\partial}{\partial x^\mu} = N_s \frac{\partial}{\partial t} \tag{97}$$

$$U_c^\mu \frac{\partial}{\partial x^\mu} = N_c \left(\frac{1-\frac{2m}{r}}{f(r)}\frac{\partial}{\partial t} + \sqrt{1-\frac{2m}{r}}\sqrt{A^2r^2 - 2ma^2r}\frac{\partial}{\partial r} \right) \tag{98}$$

Here in the last equality we have used [94] and factors N_s and N_c follow from the normalization condition:

$$-1 = g_{\mu\nu}U_s^\mu U_s^\nu \tag{99}$$

$$N_s = \frac{1}{\sqrt{f(r)}} \tag{100}$$

$$N_c = \frac{1}{\sqrt{1-\frac{2m}{r}}} \tag{101}$$

Combining these with [97-98] and [96] we get:

$$v = \sqrt{\frac{A^2r^2 - 2a^2mr}{1-\frac{2m}{r}}} \tag{102}$$

From [102] we read that v tends to 1 if one of the two horizons defined by $f(r) = 0$ is approached. This is clear because on the horizons the worldlines of the static observers become lightlike. Between the two horizons, v is decreasing from 1 to a local minimum and then increasing again to 1. It is also important to notice that this co-moving observer is a generalization of the co-moving observer used by Perlick et al. (2018) as it recreates the exact results of the above mentioned work as $a \to 0$. In Fig. 5 we can see the relative velocity curve as a function of r.

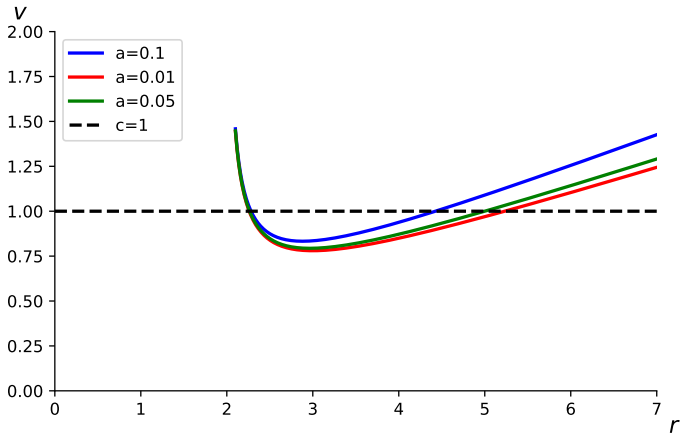

Figure 5: This plot describes relative velocity of a co-moving observer with respect to the static observer for three different values of acceleration parameter. For other parameters we have chosen: $m = 1$, $\sqrt{\Lambda/3} = 0.15$.

4.3 Shadow of the Black Hole Seen by a Co-moving Observer in the $\theta = \pi/2$ Plane

The final step of our calculations is to find the angular size of the shadow of an accelerated black hole seen by a co-moving observer in the $\theta = \pi/2$ plane. For this, we are combining [102] and [79-80]. Unfortunately the final expression is very lengthy so we will not present it here. Instead, we are showing our results through the following graph.

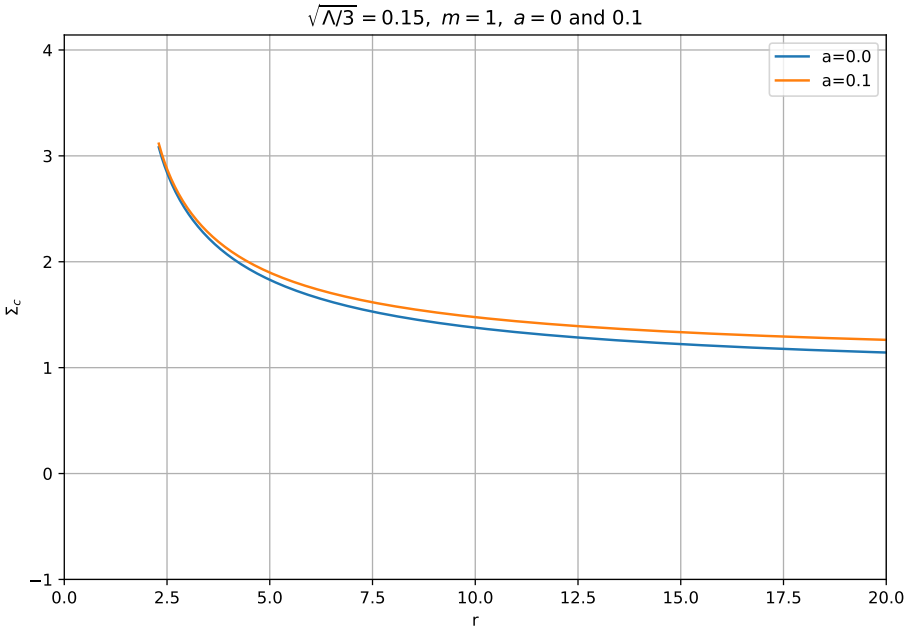

Figure 6: This plot describes the angular size of the shadow of an accelerated black hole seen by the co-moving observer at different radial coordinates. The orange curve represents the shadow size variation as the acceleration parameter is 0.1 and the blue curve represents the case when $a = 0$ - this is a well known Kottler case. Other parameters in this plot are: $m = 1, \sqrt{\Lambda/3} = 0.15$.

Here we can notice that as $r \to \infty$, the size of the shadow does not go to 0, instead it reaches an asymptotic value. To calculate this we have to use [80]:

$$\sin^2 \Sigma_C \to \frac{54A^2m^2}{1 + \sqrt{1 + 12a^2m^2} + 6a^2m^2(3 + 2\sqrt{1 + 12a^2m^2})} \quad as \quad r \to \infty \tag{103}$$

We can also see the results of [103] on the Fig. 7. We made the plot for fixed $m = 1$ and $\sqrt{\Lambda/3} = 0.15$ parameters. Because of [6] we are only varying the acceleration parameter in the range $[0, 0.5)$. We see that when $a = 0$ the value that is approached is given by the formula:

$$sin^2\Sigma_C = 9\Lambda^2 \tag{104}$$

Which exactly matches the angular size of the shadow of the Kottler black hole observed by a co-moving observer calculated by Perlick et al. (2018).

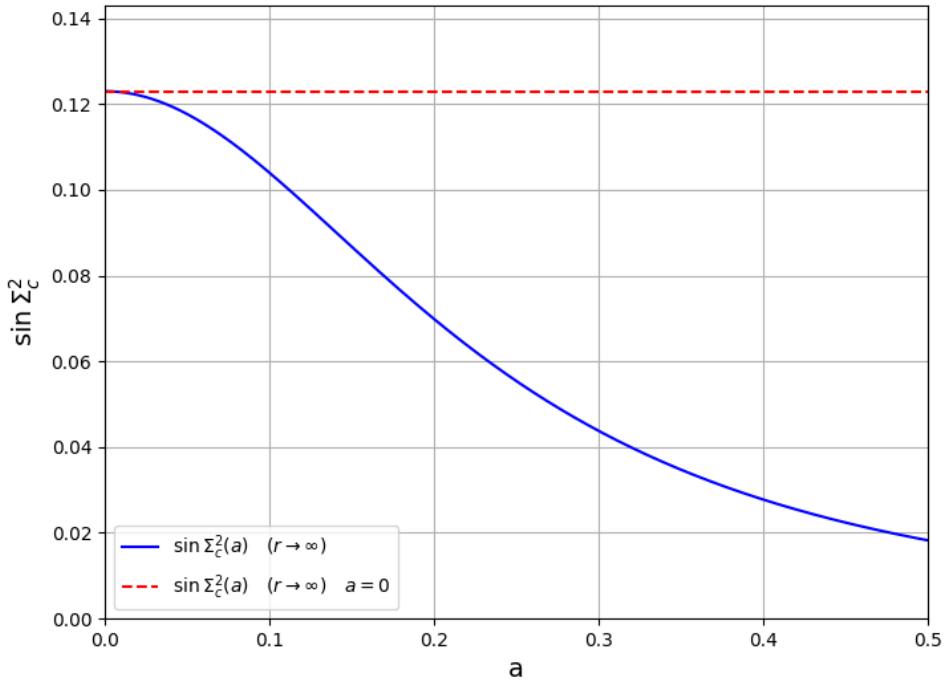

Figure 7: This describes how the acceleration parameter effects the angular radius of the shadow of the black hole as $r \to \infty$. Other parameters have following values in this plot: $m = 1, \sqrt{\Lambda/3} = 0.15$.

We can also see from [79] that:

$$\sin^2 \Sigma_C \to 0 \quad as \quad r \to r_{bh} \tag{105}$$

When the comoving observer starts at the inner horizon, the shadow covers the entire sky, $\Sigma_C = \pi$ On his way out to infinity, the shadow monotonically shrinks to a finite value given by [103], see Fig. 6. Nothing particular happens when the observer crosses the outer horizon.

Conclusion

In this thesis, we have analyzed the shadow of an accelerated black hole embedded in a de Sitter background, using the C-metric as an exact solution of Einstein's equations. We began by examining the properties of the metric, identifying the relevant parameter ranges and the physical region of interest bounded by the inner and outer horizons. We then studied the propagation of light in this spacetime, deriving the null geodesic equations and characterizing the photon sphere that defines the boundary of the black hole shadow.

For a static observer, we constructed a local tetrad frame and determined the angular radius of the shadow by relating the critical constants of motion of null geodesics to the observer's sky. The shadow was found to be circular, with its size dependent on the mass, acceleration, and cosmological constant. Given by:

$$\sin\Sigma_{ph} = \frac{\sqrt{Q(r_O)}}{\sqrt{Q(r_{ph})}}\frac{r_{ph}^2}{r_o^2} \tag{106}$$

To explore the role of cosmological expansion and observer motion, we introduced a co-moving observer in the $\theta = \pi/2$ plane. We defined this observer via a coordinate transformation that casts the metric into a cosmological form. In this frame, we obtained the co-moving observer's four-velocity, which turned out to be:

$$v = \sqrt{\frac{A^2r^2 - 2a^2mr}{1 - \frac{2m}{r}}} \tag{107}$$

and derived the angular size of the shadow as seen by such an observer. This allowed us to investigate how the shadow varies from the perspective of an observer following the expansion of the universe, offering a richer physical interpretation than the static case alone.

We also studied that the size of the shadow approaches constant(for fixed a, m, Λ) at large distances($r \to \infty$). This is given by:

$$\sin^2\Sigma_c \to \frac{54A^2m^2}{1+\sqrt{1+12a^2m^2}+6a^2m^2(3+2\sqrt{1+12a^2m^2})} \quad as \quad r\to\infty \tag{108}$$

The results presented here extend the study of black hole shadows beyond asymptotically flat and purely static spacetimes, highlighting the combined influence of acceleration and cosmological expansion on observable features. While uniformly accelerated black holes remain hypothetical, their analysis deepens our understanding of exact solutions in general relativity and illustrates how strong-field phenomena can be influenced by global properties of the universe. Future work should generalize this study to co-moving observers located off the $\theta = \pi/2$ plane, thereby providing a complete description of the angular dependence of the shadow in cosmological settings. Such investigations will further our ability to connect theoretical predictions with potential observational signatures of nontrivial spacetimes.

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